1. (25 pts) Give formal definitions to the statement "$(X, d)$ *is a metric space*".

A set $X$ (or ordered pair $(X,d)$) is called a metric space if $\exists$ a metric $d: X \times X \to \mathbb{R}$ s.t. $\forall p, q \in X$,

(1) $d(p,q) \geq 0$ $(= 0 \Leftrightarrow p = q)$

(2) $d(p,q) = d(q,p)$

(3) $\forall w \in W$, $d(p,w) + d(q,w) \geq d(p,q)$.

2. (25 pts) Let $X = [0, \infty)$. Is $d(x, y) = \left(\sqrt{x} - \sqrt{y}\right)^2$ a metric on $X$? Prove or disprove.

The answer is no, because

$$d(x,z) + d(z,y) \geq d(x,y)$$

$$\Leftrightarrow (\sqrt{x} - \sqrt{z})^2 + (\sqrt{z} - \sqrt{y})^2 \geq (\sqrt{x} - \sqrt{y})^2$$

$$\Leftrightarrow z + \sqrt{xy} \geq \sqrt{xz} + \sqrt{yz}$$

$$\Leftrightarrow (\sqrt{z} - \sqrt{x})(\sqrt{z} - \sqrt{y}) \geq 0$$

which does not hold when $x > z > y$ or $y > z > x$.

Hence $d$ is not a metric on $X$.

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3. (15 pts each) Countable or Uncountable? Explain it in a few lines. (No need to be too rigorous, just to make sure you are not guessing.)

(i) The set of irrational numbers.

(ii) The set of infinite sequences with terms = 0 or 1.

(i) Since $\mathbb{R}$ is uncountable and $\mathbb{Q}$ is countable, we have $\{\text{irrational numbers}\} = \mathbb{R} \setminus \mathbb{Q}$ is an uncountable set.

(ii) Denote the set by $S$. Suppose $S$ is countable, then $\exists\, \mathbb{N} \xrightarrow{f} S$, and $f(n)$ is the sequence $(x_1^n, x_2^n, x_3^n, \dots)$, $x_i^n \in \{0,1\}$. Let $y = (y_1, y_2, \dots)$ and $y_j \in \{0,1\}$, $y_j \neq x_j^j$ $\forall j \in \mathbb{N}$, then there's no $n \in \mathbb{N}$ s.t. $f(n) = y$ from our construction $(\rightarrow\leftarrow)$.

Hence $S$ is uncountable. #

4. (30 pts) Prove that $(\mathbb{R}^n, d)$ is a metric space for $n \in \mathbb{N}$, where $d$ denotes the Euclidean metric.

$\forall\, p, q, w \in \mathbb{R}^n$, suppose $p = (x_1, \dots, x_n)$, $q = (y_1, \dots, y_n)$, $w = (z_1, \dots, z_n)$.

- $d(p,q) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \geq 0$, and $d(p,q) = 0 \iff x_i = y_i \ \forall 1 \leq i \leq n$
$\iff p = q$.

- $d(p,q) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2} = d(q,p)$.

- $$\begin{aligned}
(d(p,w) + d(w,q))^2 &= \left(\sqrt{\sum_{i=1}^{n} (x_i - z_i)^2} + \sqrt{\sum_{i=1}^{n} (z_i - y_i)^2}\right)^2 \\
&= \sum_{i=1}^{n} (x_i - z_i)^2 + \sum_{i=1}^{n} (z_i - y_i)^2 \\
&\quad + 2\sqrt{\sum_{i=1}^{n} (x_i - z_i)^2 \sum_{i=1}^{n} (z_i - y_i)^2} \\
&\geq \sum_{i=1}^{n} (x_i - z_i)^2 + \sum_{i=1}^{n} (z_i - y_i)^2 \\
&\quad + 2\sum_{i=1}^{n} |x_i - z_i| \cdot |z_i - y_i| \\
&= \sum_{i=1}^{n} (|x_i - z_i| + |z_i - y_i|)^2 \\
&\geq \sum_{i=1}^{n} (x_i - y_i)^2 \\
&= d(p,q)^2
\end{aligned}$$

by Cauchy-Schwarz inequality

Since $d(p,q)$, $d(p,w)$, $d(w,q) \geq 0$, we conclude that
$$d(p,w) + d(w,q) \geq d(p,q).$$

Hence $\mathbb{R}^n$ with Euclidean metric is a metric space $\forall n \in \mathbb{N}$. #

5. (28 pts) Prove that the set of all polynomials

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

with integral coefficients is countable. Deduce the set of algebraic numbers is countable. (An algebraic number is a number which is a root of a polynomial with integral coefficients.)

Let $P$ denote the set of all polynomials with integral coefficients, and $P_n$ denotes the set of polynomial of degree $\leq n$ in $P$.

Claim: $P_n \simeq \mathbb{Z}^{n+1}$

subpf. $f: P_n \longrightarrow \mathbb{Z}^{n+1}$ is bijective.

$a_n x^n + \cdots + a_0 \longmapsto (a_n, a_{n-1}, \ldots, a_0)$

Back to original problem, by our claim,

$$P = \bigcup_{n \in \mathbb{N}_0} P_n \simeq \bigcup_{n \in \mathbb{N}_0} \mathbb{Z}^{n+1}$$

which is a countable union of countable sets, so we have $P$ is countable. Since $P$ is countable, we have $\{\text{argebric numbers}\} = \bigcup_{p \in P} \{\text{roots of } p\}$ is a countable union of countable sets, and hence countable. #