Introduction to Real Analysis, Quiz 4

1. (25 pts) Give formal definitions to the statement "(X, d) *is a metric space*".
2. (25 pts) Let $X = [0, \infty)$. Is $d(x, y) = \left(\sqrt{x} - \sqrt{y}\right)^2$ a metric on X? Prove or disprove.
3. (15 pts each) Countable or Uncountable? Explain it in a few lines. (No need to be too rigorous, just to make sure you are not guessing.)

   (i) The set of irrational numbers.

   (ii) The set of infinite sequences with terms = 0 or 1.
4. (30 pts) Prove that $(\mathbb{R}^n, d)$ is a metric space for $n \in \mathbb{N}$, where d denotes the Euclidean metric.
5. (28 pts) Prove that the set of all polynomials

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

   with integral coefficients is countable. Deduce the set of algebraic numbers is countable. (An algebraic number is a number which is a root of a polynomial with integral coefficients.)