Introduction to Real Analysis, Quiz 3 answer key

1. State and prove *the Cauchy-Schwarz inequality.*

   Solution. For $\vec{a}, \vec{b} \in \mathbb{C}^n$,

$$\left|\langle \vec{a}, \vec{b}\rangle\right|^2 \leq \langle \vec{a}, \vec{a}\rangle\langle \vec{b}, \vec{b}\rangle,$$

   or

$$\left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \leq \sum_{j=1}^{n} \left|a_j\right|^2 \sum_{j=1}^{n} \left|b_j\right|^2.$$

   The proof: For $\mathbb{C}^n$, if $\vec{b} = 0$, then it is done, else for any $x \in \mathbb{C}$, consider the function

$$\begin{aligned} 0 \leq |\vec{a} - x\vec{b}|^2 &= \langle \vec{a} - x\vec{b}\rangle \\ &= \langle \vec{a}, \vec{a}\rangle - \langle \vec{a}, x\vec{b}\rangle - \langle x\vec{b}, \vec{a}\rangle + \langle x\vec{b}, x\vec{b}\rangle \\ &= \langle \vec{a}, \vec{a}\rangle - \bar{x}\langle \vec{a}, \vec{b}\rangle - x\langle \vec{b}, \vec{a}\rangle + x\bar{x}\langle \vec{b}, \vec{b}\rangle \end{aligned}$$

   Now, we set $x = \dfrac{\langle \vec{a}, \vec{b}\rangle}{\langle \vec{b}, \vec{b}\rangle}$ we will get

$$\begin{aligned} 0 &\leq \langle \vec{a}, \vec{a}\rangle - \frac{|\langle \vec{a}, \vec{b}\rangle|^2}{\langle \vec{b}, \vec{b}\rangle} \\ \Longrightarrow \left|\langle \vec{a}, \vec{b}\rangle\right|^2 &\leq \langle \vec{a}, \vec{a}\rangle\langle \vec{b}, \vec{b}\rangle \end{aligned}$$

   Note that $\langle \vec{b}, \vec{a}\rangle = \overline{\langle \vec{a}, \vec{b}\rangle}$. ■

2. Let $z_1, z_2 \cdots, z_n$ be complex numbers, prove that

$$|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$$

   Hint. Use Induction and prove the base case as detailed as you can.

Solution. Following the hint, we consider $n = 2$ first.

$$\begin{aligned}
|z+w| &= \sqrt{(z+w)(\overline{z+w})} \\
&= \sqrt{z\bar{z} + w\bar{z} + z\bar{w} + w\bar{w}} \\
&= \sqrt{|z|^2 + 2\mathrm{Re}(zw) + |w|^2} \\
&\leq \sqrt{|z|^2 + 2|zw| + |w|^2} \\
&= \sqrt{(|z| + |w|)^2} \\
&= |z| + |w|.
\end{aligned}$$

Now, for the general n,

$$\begin{aligned}
|z_1 + \cdots, +z_n| &= |(z_1 + (z_2 + \cdots + z_n)| \\
&\leq |z_1| + |z_2 + \cdots + z_n| \\
&= |z_1| + |z_2 + (z_3 + \cdots + z_n)| \\
&\leq |z_1| + |z_2| + |z_3 + \cdots + z_n| \\
&\leq \cdots \\
&\leq |z_1| + \cdots + |z_n|.
\end{aligned}$$

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3. Prove the following statement, "Principle of Induction $\Rightarrow$ Well-Ordering Principle."

Solution. Recall:

- Principle of Induction:
  Let S be a subset if $\mathbb{N}$, such that
  - $1 \in S$
  - If $k \in S$, then $k+1 \in S$.

  Then $S = \mathbb{N}$.
- Well-Ordering Principle: Any non-empty subset of $\mathbb{N}$ has a least element.

POI $\Longrightarrow$ WOP:
We prove by contradiction, assume S is a subset of $\mathbb{N}$ with no least element. We know that $1 \notin S$ because S has no least element. Since $1 \notin S$, $2 \notin S$. By this argument, we get if $a \notin S$ for all $a \nleq k$, then $k+1 \notin S$.
Now consider the set $\mathbb{N} \setminus S$. We know the set satisfies the condition that

- $1 \in \mathbb{N} \setminus S$

- If $k \in \mathbb{N} \setminus S$, then $k+1 \in \mathbb{N} \setminus S$,

which implies $\mathbb{N} \setminus S = \mathbb{N}$ and S is an empty set, contradicting that S is non-empty. Hence the statement is correct. ■

4. Let $z = a + ib, w = u + iv$ and $z^2 = w$. Calculate a, b in terms of u, v. (Reminder. There are two roots.)

Solution. Expand z^2 and we get

$$z^2 = (a+bi)^2 = a^2 + 2abi - b^2 = w = u + vi$$

Solve the following two equations

$$\begin{cases} a^2 - b^2 = u \\ 2ab = v \end{cases}$$

$$(a = \frac{v}{2b}) \implies (\frac{v}{2b})^2 - b^2 = u$$

Solve the two equations and obtain

$$a^2 = \frac{u + \sqrt{u^2+v^2}}{2}, \; b^2 = \frac{-u + \sqrt{u^2+v^2}}{2}.$$

(Since a^2, b^2 are positive.) Hence the roots of a and b is: If $v \geq 0$,

$$a = \pm\sqrt{\frac{u + \sqrt{u^2+v^2}}{2}}, \; b = \pm\sqrt{\frac{-u + \sqrt{u^2+v^2}}{2}}$$

and if $v \leq 0$,

$$a = \pm\sqrt{\frac{u + \sqrt{u^2+v^2}}{2}}, \; b = \mp\sqrt{\frac{-u + \sqrt{u^2+v^2}}{2}}.$$

■

5. Suppose z is a complex number with $|z| = 1$, calculate

$$|1+z|^2 + |1-z|^2,$$

and interpret it geometrically. (Hint. What is the geometric interpretation of $|a - b|$?)

Solution. Suppose $z = a + bi$, $|z| = 1$ implies $a^2 + b^2 = 1$. Calculate

$$\begin{aligned} |1+z|^2 + |1-z|^2 &= |(a+1) + bi|^2 + |(1-a) - bi|^2 \\ &= (a+1)^2 + b^2 + (1-a)^2 + (-b)^2 \\ &= a^2 + 2a + 1 + b^2 + 1 - 2a + a^2 + b^2 \\ &= 2(a^2+b^2) + 2 = 4. \end{aligned}$$

The geometrical explain is that, z is on the unit circle of the complex plane, and $|1+z|^2 + |1-z|^2$ measures the distance squared between z and -1 plus the distance squared between z and 1. And, by the common sense of right triangle, this value is always 4. ■