1. Give formal definitions to the following statements.

(i) a is the *least upper bound* of the set $S \subset \mathbb{R}$.

(ii) $S \subset \mathbb{R}$ satisfies *the least upper bound property.*

(i) a is called the least upper bound of the set $S \subset \mathbb{R}$ if

(1) a is an upper bound of S.

(2) If $\gamma < a$, then γ is not an upper bound of S.

(ii) S is said to satisfy the least upper bound property if every non-empty subset of S having upper bound has a least upper bound.

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2. Let $A = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$. What are $\sup A, \inf A$?

- $\sup A = 1$: 1 is u.b. of A since $\forall n \in \mathbb{N}$, $1 \geq \frac{1}{n}$.

Moreover, since $1 \in A$, every $x < 1$ is not u.b. of A.

Hence $\sup A = 1$.

- $\inf A = 0$: 0 is l.b. of A since $\forall n \in \mathbb{N}$, $0 \leq \frac{1}{n}$.

On the other hand, $\forall x \in \mathbb{R}^+$, by Archemedean property, $\exists n \in \mathbb{N}$ s.t. $nx > 1$, i.e., $x > \frac{1}{n}$, so x is not l.b. of A. Hence $\inf A = 0$. #

3. For $E \subseteq \mathbb{R}$, prove that

$$\inf E = -\sup(-E).$$

Suppose $\inf E = \gamma$. It suffices to show $\sup(-E) = -\gamma$.

From definition, we know that

(1) γ is l.b. of E, i.e., $\forall x \in E$, $\gamma \le x$.

(2) $\forall y > \gamma$, y is not l.b. of E.

Notice that (1) implies that $\forall x \in E$, $-\gamma \geq -x$, so $-\gamma$ is u.b. of $-E$; (2) says that $\forall y > \gamma$, $\exists x \in E$ s.t. $y > x$. Hence $\forall -y < -\gamma$, $\exists -x \in -E$ s.t. $-y < -x$, i.e. $-y$ is not u.b. of $-E$. We therefore conclude that $\sup(-E) = -\gamma$. #

4. Let $a > 1$. We assume that $a^{1/n}$ is already a well-defined notion in the following context for $n \in \mathbb{N}$, which denotes the unique positive solution of $x^n = a$.

(i) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that

$$(a^m)^{\frac{1}{n}} = (a^p)^{\frac{1}{q}}.$$

(ii) Prove that $a^{r+s} = a^r a^s$ if r and s are rational.

(iii) If x is real, define $A(x)$ to be the set of all numbers a^t, where t is rational and $t \leq x$. Prove that

$$a^r = \sup A(r)$$

when r is rational. Hence it makes sense to define

$$a^x = \sup A(x)$$

for every real x.

(iv) Prove that $a^x a^y = a^{x+y}$ for all $x, y \in \mathbb{R}$.

(i) We have $\left((a^m)^{\frac{1}{n}}\right)^{nq} = \left(\left((a^m)^{\frac{1}{n}}\right)^n\right)^q = (a^m)^q = a^{mq}$,

$\left((a^p)^{\frac{1}{q}}\right)^{nq} = \left(\left((a^p)^{\frac{1}{q}}\right)^q\right)^n = (a^p)^n = a^{pn} = a^{mq}$

by $\frac{m}{n} = \frac{p}{q}$. Hence $(a^m)^{\frac{1}{n}} = (a^{mq})^{\frac{1}{nq}} = (a^p)^{\frac{1}{q}}$. (By the uniqueness of the solution to $x^n = a$).

(ii) Let $r = \frac{m}{n}$, $s = \frac{p}{q}$, then from (i),

$(a^{r+s})^{nq} = \left(a^{\frac{mq+pn}{nq}}\right)^{nq} = \left(\left(a^{\frac{1}{nq}}\right)^{mq+pn}\right)^{nq} = a^{mq+pn}$

$(a^r \cdot a^s)^{nq} = (a^r)^{nq} \cdot (a^s)^{nq} = \left((a^r)^n\right)^q \cdot \left((a^s)^q\right)^n$

$= a^{mq} \cdot a^{pn} = a^{mq+pn} = (a^{r+s})^{nq}$

By uniqueness of n-th root again, $a^{r+s} = a^r \cdot a^s$.

(Note that $(a^{\frac{m}{n}})^n = ((a^{\frac{1}{n}})^m)^n = ((a^{\frac{1}{n}})^n)^m = a^m$).

(iii) $\forall t < r$, suppose $t = \frac{m}{n}$, $r = \frac{p}{q}$, then $mq < pn$,

so $a^{mq} < a^{pn}$ $(a > 1)$. Since

$$(a^{pn} - a^{mq}) = (a^{\frac{p}{q}} - a^{\frac{m}{n}})\underbrace{((a^{\frac{p}{q}})^{nq-1} + (a^{\frac{p}{q}})^{nq-2} \cdot a^{\frac{m}{n}} + \dots + (a^{\frac{m}{n}})^{nq-1})}_{>0} > 0$$

so $a^{\frac{p}{q}} - a^{\frac{m}{n}} > 0$, i.e., $a^r > a^t$, thus a^r is

u.b. of $A(r)$. Meanwhile, since $a^r \in A(r)$, every $\gamma < a^r$

can't be u.b. of $A(r)$. Hence $\sup A(r) = a^r$.

(iv) $a^{x+y} = \sup A(x+y)$ $(\forall x, y \in \mathbb{R})$

$$= \sup\{a^t \mid t \in \mathbb{Q},\ t \le x+y\}$$

$$= \sup\underbrace{\{a^t \mid t \in \mathbb{Q},\ t < x+y\}}_{(1)}$$

The last equation holds since if $x+y \notin \mathbb{Q}$, then t can't

equal to $x+y$, so $\le$ can be replaced by $<$. When $x+y \in \mathbb{Q}$,

a^{x+y} is still u.b. of (1). $\forall x < a^r$, it suffices to find $t \in \mathbb{Q}$ s.t. $x < a^t < a^r$ so that x won't be an upper bound of (1). Notice that

$$a^r - a^t \overset{\text{by (ii)}}{=} a^t(a^{r-t}-1) \overset{\text{by (iii)}}{<} a^r(a^{r-t}-1).$$ By density property of $\mathbb{Q}$, $\forall n \in \mathbb{N}$, $\exists t_n \in \mathbb{Q}$ s.t. $r - \frac{1}{n} < t_n < r$. Also, since

$$a - 1 = (a^{\frac{1}{n}} - 1)\underbrace{\left((a^{\frac{1}{n}})^{n-1} + (a^{\frac{1}{n}})^{n-2} + \dots + 1\right)}_{>n}$$

$$> n(a^{\frac{1}{n}} - 1)$$

$$\Rightarrow (a^{\frac{1}{n}} - 1) < \frac{a-1}{n}$$

Hence we have

$$a^r - a^{t_n} < a^r(a^{r-t_n} - 1) < a^r(a^{\frac{1}{n}} - 1) < \frac{a^r(a-1)}{n}$$

This means that we can choose n large enough s.t. $0 < a^r - a^{t_n} < a^r - x$, so that $a^r > a^{t_n} > x$.

Hence x is not u.b. of (1), we have $\sup(1) = a^r$.

On the other hand,

$$a^x \cdot a^y = \sup A(x) \cdot \sup A(y)$$

$$= \sup\{a^u \mid u\in\mathbb{Q},\ u\le x\}\cdot\sup\{a^v \mid v\in\mathbb{Q},\ v\le y\}$$

Similar to previous discussion

$$= \sup\{a^u \mid u\in\mathbb{Q},\ u< x\}\cdot\sup\{a^v \mid v\in\mathbb{Q},\ v< y\}$$

$$= \sup\{a^u\cdot a^v \mid u,v\in\mathbb{Q},\ u<x,\ v<y\}$$

$\sup A\cdot\sup B = \sup AB$ if $A,B\subseteq\mathbb{R}^+$, and non-empty

$$= \sup\underbrace{\{a^{u+v} \mid u,v\in\mathbb{Q},\ u<x,\ v<y\}}_{(2)}$$

by (ii)

Notice that $\{u+v \mid u,v\in\mathbb{Q},\ u<x,\ v<y\} = \{t \mid t\in\mathbb{Q},\ t<x+y\}$ from the definition of addition of real numbers. Hence $(1)=(2)$,

$\sup(1)=\sup(2)$. #