Introduction to Real Analysis, Quiz 2

1. (30 pts, 15pts each) Give formal definitions to the following statements.

   (i) a is the *least upper bound* of the set $S \subset \mathbb{R}$.

   (ii) $S \subset \mathbb{R}$ satisfies *the least upper bound property.*

2. (30 pts, 15pts each) Let $A = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$. What are $\sup A, \inf A$?

3. (30 pts) For $E \subseteq \mathbb{R}$, prove that

$$\inf E = -\sup(-E).$$

4. Let $a > 1$. We assume that $a^{1/n}$ is already a well-defined notion in the following context for $n \in \mathbb{N}$, which denotes the unique positive solution of $x^n = a$.

   (i) (14 pts) If m, n, p, q are integers, $n > 0,\ q > 0$, and $r = m/n = p/q$, prove that

$$(a^m)^{\frac{1}{n}} = (a^p)^{\frac{1}{q}}.$$

   (ii) (10 pts) Prove that $a^{r+s} = a^r a^s$ if r and s are rational.

   (iii) (14 pts) If x is real, define $A(x)$ to be the set of all numbers a^t, where t is rational and $t \le x$. Prove that

$$a^r = \sup A(r)$$

   when r is rational. Hence it makes sense to define

$$a^x = \sup A(x)$$

   for every real x.

   (iv) (10 pts) Prove that $a^x a^y = a^{x+y}$ for all $x, y \in \mathbb{R}$.