1. (30 pts, 15pts each) Give formal definitions to the following statements.

   (a) R is a *relation* between sets A and B.

   (b) $\overset{<}{\cancel{<}}$ is an *order* on the set S.

(a) (Lecture 1) A relation R is a subset of $A \times B$.

(b) (Lecture 2) An order on set S is a relation $<$ satisfying

① If $x, y \in S$, then exactly one of the statement holds: $x < y$, $x = y$, $y < x$.

② If $x, y, z \in S$, and $x < y$, $y < z$, then $x < z$.

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2. (32 pts, 8pts each) Let $A = \{1, 2, 3\}, B = \{3, 4\}$. What are $A \cap B, A \cup B, A \setminus B, A \times B$?

$A \cap B = \{3\}$, $A \setminus B = \{1, 2\}$

$A \cup B = \{1, 2, 3, 4\}$, $A \times B = \{(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)\}$ #

3. (28 pts) Define the addition of rational numbers and check that your definition is well-defined, that is, if $\frac{a}{b} = \frac{a'}{b'}$, $\frac{c}{d} = \frac{c'}{d'}$, then $\frac{a}{b} + \frac{c}{d} = \frac{a'}{b'} + \frac{c'}{d'}$.

By definition, $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ — ①

$\frac{a'}{b'} + \frac{c'}{d'} = \frac{a'd'+b'c'}{b'd'}$ — ②

Since $\frac{a}{b} = \frac{a'}{b'}$, $ab' = a'b$

$\frac{c}{d} = \frac{c'}{d'}$, $cd' = c'd$

① = ② $\Leftrightarrow (ad+bc)b'd' = (a'd'+b'c')bd$

$\Leftrightarrow ab'dd' + bb'cd' = a'bdd' + bb'c'd.$

$\Leftrightarrow (ab' - a'b)dd' + bb'(cd' - c'd) = 0$ #

4. (28 pts) Prove that there is no rational number whose square is 12.

If $\exists p, q \in \mathbb{Z}$ s.t. $(p,q)=1$, $\left(\frac{q}{p}\right)^2 = 12$, then $2 \mid q$, $2 \nmid p$. Let $q' = \frac{q}{2}$,

then $\left(\frac{q'}{p}\right)^2 = 3$, $(p, q') = 1 \Rightarrow q'^2 = 3p^2$, $3 \mid q'$. Hence $q' = 3q''$

for some $q'' \in \mathbb{Z}$. $\Rightarrow 9q''^2 = 3p^2$, $3q''^2 = p^2 \Rightarrow 3 \mid p$, which

contradicts the fact that $(p,q) = 1$ $(\rightarrow\leftarrow)$. #

5. (20 pts) Let F be an ordered field and $0 \in F$ be the additive identity. Prove that if $x \neq 0$, then $x^2 > 0$. (Consequently, the multiplicative identity is positive. Moreover, $\mathbb{C}$ is not an ordered field since $i^2 = -1$.)

Note: You should carefully prove $0x = 0$ first if you need to use this fact.

(i) $0 \cdot x = 0$:

$\forall x \in F$, $0 \cdot x = (0+0) \cdot x = 0 \cdot x + 0 \cdot x$.

existence of additive inverse

Since $0 \cdot x + (-(0 \cdot x)) = 0$, $0 = 0 \cdot x$.

(ii) If $x > 0$ (F is an ordered field!), $x^2 = x \cdot x > 0$ by definition of ordered field.

distribution law

(iii) If $x < 0$, then $(-x) \cdot x + (-x) \cdot (-x) = (-x) \cdot (x + (-x)) = (-x) \cdot 0 = 0$ from (i), Also, since $x \cdot x + (-x) \cdot x = (x + (-x)) \cdot x = 0 \cdot x = 0$ from (i), by the uniqueness of additive inverse of $(-x) \cdot x$, $x^2 = (-x) \cdot (-x) > 0$ from (ii).

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