# Introduction to Real Analysis, Quiz 13

1. Let $\{f_n\}$ be a sequence of functions. What does one mean by "$f_n \to f$ uniformly."

   *Solution.* We say $f_n \overset{u}{\to} f$ if, $\forall \epsilon > 0$, $\exists N$ such that $n \geq N$ implies $||f_n - f|| < \epsilon$. ∎

2. Prove that the following function sequence converges pointwisely, calculate its limit and determine whether they converges uniformly.

$$f_n(x) = x^n \text{ for } x \in [0, 1]$$

   *Solution.* For $x \in [0, 1)$, $f_n(x) = x^n$ and $\lim_{n\to\infty} f_n(x) = 0$.
   For $x = 1$, $f_n(1) = 1$ and $\lim_{n\to\infty} f_n(1) = 1$.

$$f(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases} .$$

   It doesn't converge uniformly, since $f$ is not continuous. ∎

3. State *Taylor's Theorem.*

   *Solution.* If $f^{(n-1)}$ is continuous on $[a, b]$ and $f^{(n)}$ exists on $(a, b)$, then $P_{n-1}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}}{(n-1)!}(x-a)^{n-1}$ approximates $f(x)$ and $f(x) = P_{n-1}(x) + \frac{f^{(n)}(c)}{n!}(x-a)^n$ for some $c \in (a, b)$. ∎

4. Prove that $f_n(x) = \sum_{k=0}^{n} \frac{x^n}{n!} f^{(n)}(0)$ converges uniformly to $f(x)$ on $[0, 100]$ for $f(x) = \sin x$.

   *Solution.* By Taylor's theorem, we have $f(x) - f_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^n$ for some $c \in [0, 100]$. Therefore, we consider $|f_n(x) = f| = \sup |\frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1}| \leq \frac{100^{n+1}}{(n+1)!}$, which converges to 0. Hence it converges uniformly. ∎

5. Suppose $a \in \mathbb{R}$, and $f$ is twice differentiable on $(a, \infty)$. Let $|f(x)|, |f'(x)|, |f''(x)|$ be bounded and $M_0, M_1, M_2$ are their least upper bounds respectively. Prove that

$$M_1^2 \leq 4M_0M_2.$$

Hint. Use Taylor's Theorem to prove, if $h > 0$,

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi)$$

for some $\xi \in (x, x+2h)$. Hence

$$M_1 \leq \frac{M_0}{h} + hM_2.$$

Pick appropriate $h$.

*Solution.* Following the hint, for $x \in [a, \infty)$ and $h > 0$, Taylor's theorem would imply

$$\begin{aligned} f(x+2h) &= f(x) + f'(x)(x+2h-x) + \frac{f''(\zeta)}{2!}(x+2h-x)^2 \\ &= f(x) + f'(x)2h + f''(\zeta)2h^2 \\ \implies f'(x) &= \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\zeta) \end{aligned}$$

for some $\zeta \in (x, x+2h)$. Now, we let $h = \sqrt{\frac{M_0}{M_2}}$,

$$\begin{aligned} & f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\zeta) \\ \implies & M_1 \leq \frac{1}{h}M_0 + hM_2 \\ \implies & M_1 \leq 2\sqrt{M_0M_2} \\ \implies & M_1 \leq 4M_0M_2. \end{aligned}$$

■