Quiz 11 Answer Key

1. State the *Intermediate Value Theorem.*

   Solution. If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < c < f(b)$, then there exists x such that $f(x) = c$.
   (proof). Since $[a, b]$ is connected and f is continuous, $f([a, b])$ is connected. Therefore, if we cannot find x such that $f(x) = c$, then the sets $[\inf f([a, b]), c)$ and $(c, \sup f([a, b])]$ separate $f([a, b])$, which would be a contradiction. ■

2. Let $f : X \to Y$ be a continuous function between two metric spaces. Prove that $f^{-1}(F)$ is closed in X if F is closed in Y.

   Solution. We know that f is continuous if and only if, for all open set $\mathcal{U}$ in Y, $f^{-1}(\mathcal{U})$ is open in X. Therefore, if F is closed in Y, then F^c is open in Y, we have $f^{-1}(F^c) = (f^{-1}(F))^c$ is open in X. Hence $f^{-1}(F)$ is closed in X. ■

3. Let $f : X \to Y$, $g : Y \to Z$ be continuous functions between metric spaces. Show that $g \circ f$ is continuous.

   Solution. Again, note that f is continuous if and only if, for all open set $\mathcal{U}$ in Y, $f^{-1}(\mathcal{U})$ is open in X. Therefore, for any open set $\mathcal{V}$ in Z, $g^{-1}(\mathcal{V})$ is open in Y since g is continuous. Consequently, $f^{-1}(g^{-1}(\mathcal{V}))$ is open in X since f is continuous. And we now have, for any open set $\mathcal{V}$ in Z, $f^{-1}(g^{-1}(\mathcal{V})) = (g \circ f)^{-1}(\mathcal{V})$ is open in X. Hence $g \circ f$ is continuous. ■

4. Describe "continuous function preserves compactness" formally and prove it.

   Solution. The function $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.
   (proof). Let $\{\mathcal{V}_\alpha\}$ be an open covering of $f(X)$. Let $\{\mathcal{U}_\alpha\} = \{f^{-1}(\mathcal{V}_\alpha)\}$, which is an open covering of X since f is continuous. Since X is compact, there exists a finite subcovering $\mathcal{U}_{\alpha_1}, \ldots, \mathcal{U}_{\alpha_n}$. Then $\mathcal{V}_{\alpha_1}, \ldots, \mathcal{V}_{\alpha_n}$ cover $f(X)$. Hence $f(X)$ is compact. ■

5. Let $f, g : X \to Y$ be two continuous functions. Suppose that $g(x) = f(x)$ for $x \in E$, where E is dense in X. Prove that $g(x) = f(x)$ for all $x \in X$.

Solution. Let $h(x) = f(x) - g(x)$ be a continuous function such that $h(x) = 0$ for $x \in E$. By the definition of continuity, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d(x, y) < \delta$ would imply $d(f(x), f(y)) < \epsilon$. Now, if there is a point y such that $h(y) \neq 0$, we say $d(h(y), 0) = c$. Let $\epsilon = c/2$, for every $\delta > 0$, there exist some point $x \in N_\delta(y)$ and x is also in E since E is dense in X, then $d(h(y), h(x)) = d(h(y), 0) = c > \epsilon$. That would result in a contradiction. Hence $h(x) = 0$ for all $x \in X$, which implies $f(x) = g(x)$ for all $x \in X$. ■