1. Give formal definitions to the following statements.

(i) $\lim_{x\to p} f(x) = L$, in metric space (X, d).

(ii) $f : \mathbb{R} \to \mathbb{R}$ is continuous at 3.

(i) $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in X$, if $0 < d_X(x, p) < \delta$, then $d_Y(f(x), L) < \varepsilon$. $(f : X \to Y)$

(ii) $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$, if $|x-3| < \delta$, then $|f(x) - f(3)| < \varepsilon$.

#

2. Prove that the function is continuous at all $x \in \mathbb{R}$.

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- f is continuous at $x \neq 0$: Since $\frac{1}{x}$, $\sin x$ are continuous when $x \neq 0$, $\sin\frac{1}{x}$ is continuous when $x \neq 0$. Combining with x is continuous $\Rightarrow$ $x\sin\frac{1}{x}$ is continuous when $x \neq 0$.

- f is continuous at $x = 0$: $\forall \varepsilon > 0$, let $\delta = \varepsilon$, then $\forall y$ s.t. $|y - x| < \delta$, $|y\sin(\frac{1}{y}) - 0| \leq |y| < \varepsilon$.

#

3. (25 pts) Suppose $\sum_{i=0}^{\infty} a_i, \sum_{i=0}^{\infty} b_i$ converges absolutely. Let $c_n = \sum_{i=0}^{n} a_i b_{n-i}$, prove that $\sum c_n$ converges.

Suppose $\sum_{i=0}^{\infty} |a_i| \leq A$, $\sum_{i=0}^{\infty} |b_i| \leq B$. Then

$$\sum_{i=0}^{n} |c_i| = \sum_{i=0}^{n} \left| \sum_{j=0}^{i} a_i b_{j-i} \right| \leq \sum_{i=0}^{n} \sum_{j=0}^{i} |a_i b_{j-i}| =$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{i} |a_i| |b_{j-i}|$$

$$\leq \left(\sum_{i=0}^{n} |a_i| \right) \cdot \left(\sum_{i=0}^{n} |b_i| \right)$$

$$\leq A \left(\sum_{i=0}^{n} |b_i| \right) \leq AB$$

Since $\sum_{i=0}^{n} |C_n|$ bounded and is increasing, it converges.

Hence $\sum_{i=0}^{n} C_n$ converges. #

4. Prove that $\lim_{x\to p} f(x) = L$ if and only if for all sequence $\{p_n\}$ with $p_n \neq p$ and $p_n \to p$, we have $\lim_{n\to\infty} f(p_n) = L$.

- ($\Rightarrow$) If $\lim_{x\to p} f(x) = L$, then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $0 < d_X(x,p) < \delta$, $d_Y(f(x), L) < \varepsilon$. Thus $\forall$ seq. $\{p_n\}$ with $p_n \neq p$, $p_n \to p$, $\exists N$ s.t. $\forall n > N$, $0 < d_X(p_n, p) < \delta$, so $d_Y(f(p_n), L) < \varepsilon$,

  $\Rightarrow \lim_{n\to\infty} f(p_n) = L$.

- ($\Leftarrow$) Suppose $\lim_{x\to p} f(x) \neq L$. Then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists x$ s.t. $0 < d_X(x,p) < \delta$ but $d(f(x), f(p)) \geq \varepsilon$.

  Take $\delta_n = \frac{1}{n}$ and let x_n be the corresponding x s.t. $d_X(x_n, p) < \delta_n$ but $d(f(x_n), f(p)) \geq \varepsilon$.

  Then $x_n \neq p$ and $x_n \to p$. However, $\lim_{n\to\infty} f(x_n) \neq L$, ($\to\leftarrow$). Hence $\lim_{x\to p} f(x) = L$. #

5. (23 pts) Discuss the convergence of the series $\sum_{n=1}^{\infty} nr^n$ and calculate it if the limit exists. (Be careful if you need to rearrange terms.)

$\sum_{n=1}^{\infty} nr^n$ converges $\Rightarrow |nr^n| \to 0$ as $n \to \infty \Rightarrow |r| < 1$.

$\forall |r|<1$, $\exists s$ s.t. $|r|<s<1$, and since

$$\sum_{n=1}^{\infty} ns^n = \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} s^n = \sum_{k=1}^{\infty} s^k \cdot \frac{1}{1-s} = \frac{s}{(1-s)^2} < \infty,$$

$\sum_{n=1}^{\infty} nr^n$ converges absolutely, so rearranging terms is valid. Hence $\sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2}$. Also, $\forall |r| \geq 1$, the series diverges. #