Introduction to Real Analysis, Quiz 10

1. (20 pts each) Give formal definitions to the following statements.

   (i) $\lim_{x \to p} f(x) = L$, in metric space (X, d).

   (ii) $f : \mathbb{R} \to \mathbb{R}$ is continuous at 3.

2. (25 pts) Prove that the function is continuous at all $x \in \mathbb{R}$.

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

3. (25 pts) Suppose $\sum_{i=0}^{\infty} a_i$, $\sum_{i=0}^{\infty} b_i$ converges absolutely. Let $c_n = \sum_{i=0}^{n} a_i b_{n-i}$, prove that $\sum_{n=0}^{\infty} c_n$ converges.

4. (25 pts) Prove that $\lim_{x \to p} f(x) = L$ if and only if for all sequence $\{p_n\}$ with $p_n \neq p$ and $p_n \to p$, we have $\lim_{n \to \infty} f(p_n) = L$.

5. (23 pts) Discuss the convergence of the series $\sum_{n=1}^{\infty} n r^n$ and calculate it if the limit exists. (Be careful if you need to rearrange terms.)