# Introduction to Real Analysis, Quiz 13

1. (30 pts) Let $\{f_n\}$ be a sequence of functions. What does it mean for "$f_n \to f$ uniformly"?
2. (30 pts) Prove that the following function sequence converges pointwisely, calculate its limit and determine whether it converges uniformly.
$$f_n(x) = x^n,\ \ x \in [0, 1]$$
3. (30 pts) State *Taylor's Theorem.*
4. (24 pts) Prove that $f_n(x) = \displaystyle\sum_{k=0}^{n} \frac{x^n}{n!} f^{(n)}(0)$ converges uniformly to $f(x)$ on $[0, 100]$ for $f(x) = \sin x$.
5. (24 pts) Suppose $a \in \mathbb{R}$, and $f$ is twice differentiable on $(a, \infty)$. Let $|f(x)|, |f'(x)|, |f''(x)|$ be bounded and $M_0, M_1, M_2$ are their least upper bounds respectively. Prove that
$$M_1^2 \leq 4M_0M_2.$$
**Hint.** Use Taylor's Theorem to prove, if $h > 0$,
$$f'(x) = \frac{1}{2h}[f(x + 2h) - f(x)] - hf''(\xi)$$
for some $\xi \in (x, x + 2h)$. Hence
$$M_1 \leq \frac{M_0}{h} + hM_2.$$
Pick appropriate $h$ in terms of $M_0$,$M_2$.