24: The Derivative and the Mean Value Theorem

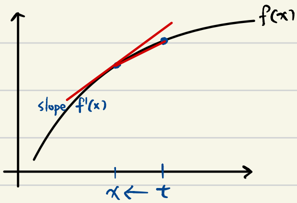
DIFFERENTIATION.

Def. A function $f: [a,b] \to \mathbb{R}$ is <u>differentiable</u> at $x \in [a,b]$ if this limit exists.

$$\boxed{f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}} \qquad t \in (a,b),\ t \neq x.$$

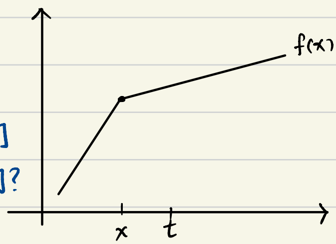
↑ The derivative of f of x.

slope of secant line



Q: If f conti on $[a,b]$ is f diff' on $[a,b]$?

$\boxed{NO}$



not diff'. at x.

If f diff' on $[a,b]$ is f conti on $[a,b]$?

$\boxed{YES}$

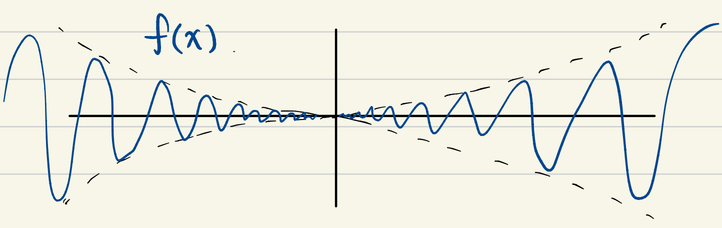
why?: $\lim_{t \to x} f(t) - f(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \cdot (t - x) = f'(x) \cdot 0 = 0$ ✓

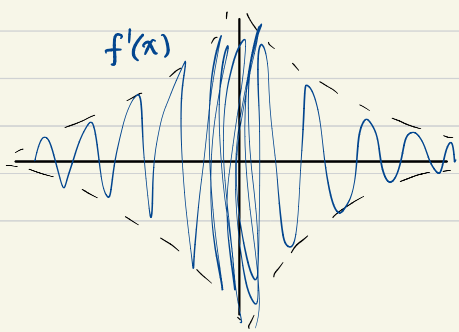
Q: If f is diff' on $[a,b]$, must f' be conti?

f' doesn't always satisfy IVP.
f' has no simple disconti.

$\boxed{NO}$

$$f(x) = \begin{cases} x^{4/3} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



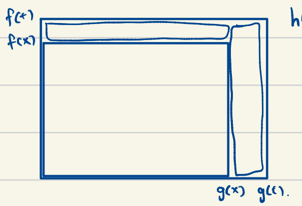


- Call a function f a $\mathcal{C}^1$ function.
  if f' exists and is conti.
  "$\mathcal{C}^k$-function" if k^{th} deri $f^{(k)}$ exists and is conti.
  $\mathcal{C}^0$-function is conti.
  C^∞ : all deri. exist.

- If f' is limit, then sum, prod, quotent rules follow.

$(f+g)' = f' + g'$ $\qquad$ $(fg)' = f'g + fg'$

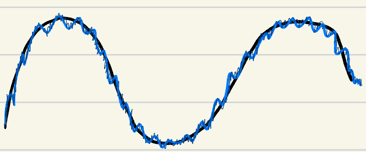
Let $h = fg$.



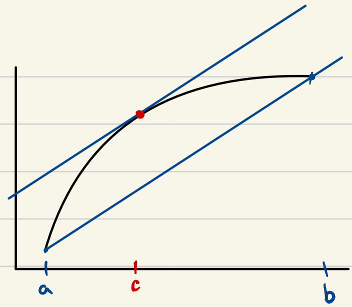
$h(t) - h(x) = f(t)\,[g(t) - g(x)]$
$\qquad + g(x)[f(t) - f(x)]$

Thm : There exists function $\mathbb{R} \to \mathbb{R}$ that are continuous everywhere, but differentiable nowhere.

Here's one : $f(x) = \displaystyle\sum_{n=1}^{\infty} b^n \cos(a^n \pi x)$ $\qquad$ $0 < b < 1$, a : odd $\in \mathbb{Z}$, $ab > 1 + \frac{3\pi}{2}$



The Mean Value Thm.



If f is conti on $[a, b]$, diff on (a, b),
then $\exists$ point $\in (a, b)$ s.t.
$$f(b) - f(a) = (b-a) \cdot f'(c).$$

- connects value of f to value of f'

Ex (appl) If $f'(x) > 0$ for all $x \in (a, b)$, then show $f(b) > f(a)$

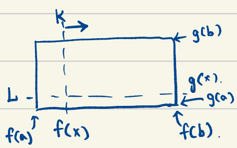
pf): $f(b) - f(a) = (b-a) \cdot f'(c) > 0$. #.

proof ① If h on $[a, b]$ has local maximum at $c \in [a, b]$
and $h'(c)$ exists $\Rightarrow h'(c) = 0$.

idea: $\underline{\dfrac{h(t) - h(c)}{t-c}} \to$ negative. $\left.\begin{array}{l} \text{negative if } t > c \quad \text{left limit} \\ \text{positive if } t < c. \quad \text{right limit} \end{array}\right\}$ exist and equal $\Rightarrow$ they must be 0.

② Generalized MVT. If $f(x), g(x)$ conti on $[a, b]$
then $\exists\, c \in (a, b)$ diff' on (a, b)
s.t. $[f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c)$.
(If $g(x) = x$, get MVT).

idea:



LHS is rate that L sweeps out area.
RHS is rate that K sweeps out area.

consider $h(x) = [f(b) - f(a)]\, g(x) - [g(b) - g(a)]\, f(x)$.

difference of area sweeps by time x.

clear: $h(a) = h(b) = 0$.

so ① $\to \exists\, c$ s.t. $h'(c) = 0$. But $h'(x) = \text{LHS} - \text{RHS}$. #.