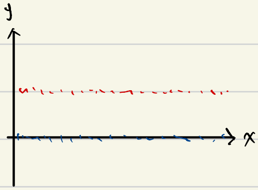
23: Discontinuous Functions

DISCONTINUOUS FUNCTIONS.

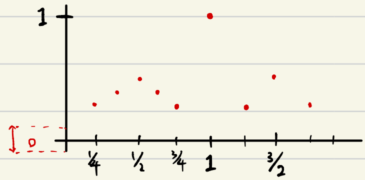
Dirichlet function.

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$



f is not continuous at any p.

Ex (HW). $f(x) = \begin{cases} \frac{1}{q} & \text{if } x = p/q \text{ least form.} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$



discontinuous at all rationals.

continuous at all irrationals.

DISCONTINUITY.

$f: (a, b) \to \mathbb{R}$

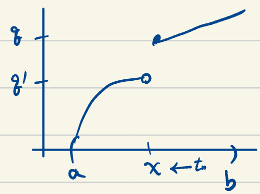
Note: for all $\{t_n\}$ in (x, b).

with $t_n \to x$, has $f(t_n) \to q$.

Write $f(x^+) = q$.

or $\lim_{t \to x^+} f(t) = q$.

Similarly, say $f(x^-) = q'$ or $\lim_{t \to x^-} f(t) = q'$

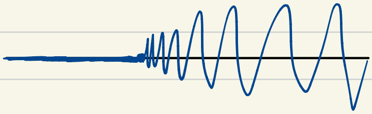


$\lim_{t \to x} f(t)$ exists $\iff$ $f(x^+) = f(x^-)$.

If f is disconti, but they exist, say f has discontinuity of first kind.

Else, second kind.

2^{nd} kind at $x=0$:

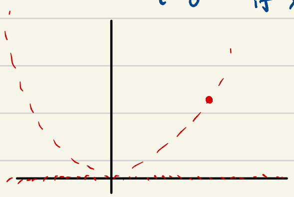


$$f(x)=\begin{cases} 0 & \text{if } x\le 0 \\ \sin(\tfrac{1}{x}) & \text{if } x>0. \end{cases}$$

<u>Ex</u> : any discontinuity of Dirichlet function is 2^{nd} kind

<u>Ex</u> : $1/q$ - Dir. func., all discontinuity is 1^{st} kind. "simple".

<u>Ex</u> $f(x)=\begin{cases} x^2 & \text{if } x\in\mathbb{Q}. \\ 0 & \text{if } x\notin\mathbb{Q}. \end{cases}$



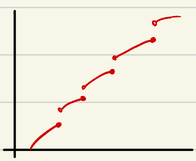
is conti at 0.

and disconti. are all 2^{nd} kind.

MONOTONE FUNCTIONS.

f : monotonely increasing if $x\le y \Rightarrow f(x)\le f(y)$.

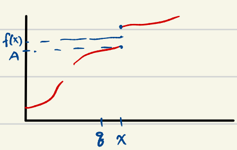
decreasing if $x\le y \Rightarrow f(x)\ge f(y)$.



<u>Thm</u> : f mono. increasing in $(a,b) \Rightarrow f(x^+),\ f(x^-)$ exist $\forall\, x, y\in(a,b)$.

In fact $\sup\limits_{t\in(a,x)} f(t) \le f(x) \le \inf\limits_{t\in(x,b)} f(t)$.

↑ bounded → exists.



call A

Claim $A=f(x^-)$

Given $\varepsilon>0$, consider $A-\varepsilon$

$\exists\,\delta$ s.t. $A-\varepsilon < f(x-\delta)\le A$. (since A is sup).

but then any $t\in(x-\delta, x)$ must satisfy $f(x-\delta)\le f(t)\le A$

so $f(t)\in(A-\varepsilon, A)$ as desired.

Similar arg. on other side.

<u>Cor</u> Mon-func. have no disconti of 2^{nd} kind.

<u>Thm</u>: f mon on (a, b)

set of pts where f is not conti is countable.

<u>pf</u>: $\forall x$ where f is disconti

pick $r(x) \in \mathbb{Q}$ s.t. $f(x^-) < r(x) < f(x^+)$

If $x, y \in D$, $r(x) \neq r(y)$. b/c f mon.

Get 1-1 <u>cor</u> between D and subset of $\mathbb{Q}$ #.