22: Uniform Continuity

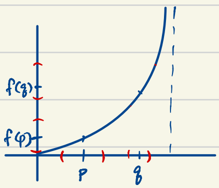
UNIFORM CONTINUITY.

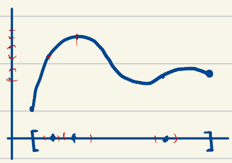
Def: Call $f: X \to Y$ uniformly continuous on X

if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$\forall x$ and p in X, $d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon$.

(same δ works for all p in X.)





Thm: $f: X \to Y$ conti, X cpt,

then f is uniformly conti on X.

(pf): Given $\varepsilon > 0$ [Goal: find a δ that works for all p].

Each pt x has δ_x ball s.t. $d(y,x) < \delta \Rightarrow d(f(y), f(x)) < \varepsilon$.

These cover X.

Lebesgue number

Q: [Can I find δ s.t. if $d(p,q) < \delta$, then p, q are in the same cover set?]

For them $d(f(p), f(q)) < d(f(p), f(x)) + d(f(q), f(x))$.

Lebesgue covering lemma: If $\{U_\alpha\}$ is open cover of cpt X.

then $\exists \delta > 0$ s.t. $\forall x \in X$, $B_\delta(x)$ is contained in some U_α.

Lebesgue # of cover.

(pf): Since X cpt, $\exists$ finite subcover $\{U_{\alpha_i}\}_{i=1}^{n}$

If K closed, define $d(x,K) = \inf_{y \in K} d(x,y)$

Claim: $d(x,K)$ is conti function of x (show).

Thm $f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, U_{\alpha_i}^c)$ is conti fun on cpt set,

so it attains its min value δ.

So if $f(x) \geq \delta$, then at least one of $d(x, U_{\alpha_i}^c) \geq \delta$, so for this i, $B_\delta(x) \subset U_{\alpha_i}$ #.

Note: $\delta > 0$, since $f(x) > 0$ at each x, since U_{α_i} are open cover.

Thm : $f: X \to Y$ conti, E connected in X, then $f(E)$ is connected.

(pf) Suppose $f(E)$ is not conn, then $f(E) = A \cup B$ a seperated union.

Note $K_A = f^{-1}(\bar{A})$, $K_B = f^{-1}(\bar{B})$ are closed (since f: conti)

Let $E_1 = f^{-1}(A) \cap E$, $E_2 = f^{-1}(B) \cap E$ } disjoint, non-empty.

$E_1 \subset K_A$ closed, so $\bar{E}_1 \subset K_A$

$E_2 \subset K_B$ closed. so $\bar{E}_2 \subset K_B$

& $K_A \cap E_2 = \phi$

($\underset{f^{-1}(\bar{A})}{\overset{\|}{K_A}}$, $\underset{f^{-1}(B)}{\overset{\cap}{E_2}}$, & $\bar{A} \cap B = \phi$.)

similarly, $K_B \cap E_1 = \phi$

So E is seperated $\times$.

Thm : Intermediate Value Thm

If $f: [a,b] \to \mathbb{R}$ conti, & $f(a) < c < f(b)$

then $\exists\, x \in (a,b)$ s.t. $f(x) = c$

(pf) $[a,b]$ conn $\Rightarrow f([a,b])$ conn.

but if c is not achieved, then "c would disconnect" $f([a,b])$.

Converse false : $f(x) = \begin{cases} 0 & , x = 0 \\ \sin(\frac{1}{x}) & , x \neq 0 \end{cases}$

not conn, but sat's IV prop.