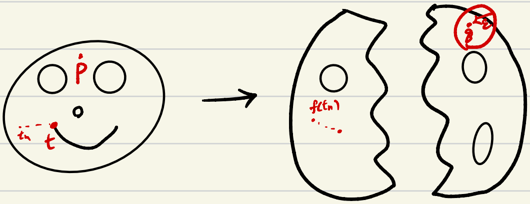
21: Continuous Functions

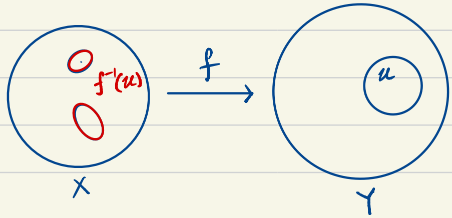
$f : X \to Y$



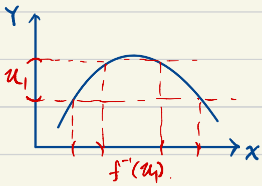
Thm : $f: X \to Y$ conti $\iff$ $\forall$ open set $\mathcal{U}$ in Y

$f^{-1}(Y)$ is open in X

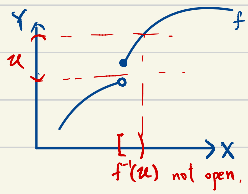
Def : $f^{-1}(\mathcal{U}) = \{x \mid f(x) \in \mathcal{U}\}$.

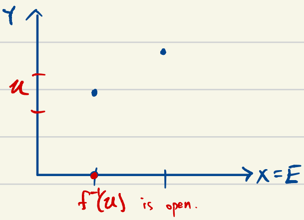


Ex



Ex





proof) ($\Rightarrow$)

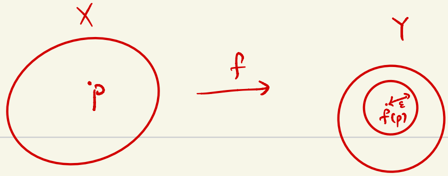
Given $\mathcal{U}$ open in X.

Consider $x \in f^{-1}(\mathcal{U})$. We'll show x is interior of $f^{-1}(\mathcal{U})$.

Note $f(x) \in \mathcal{U}$, so $\exists$ ball $N_\varepsilon(f(x)) < \mathcal{U}$.

By conti of f, $\exists$ δ-ball $N_\delta(x)$ that maps into $N_\varepsilon(f(x)) \subset \mathcal{U}$.

This means $N_\varepsilon(x) \subset f^{-1}(\mathcal{U})$, so x is interior of $f^{-1}(\mathcal{U})$.



$(\Leftarrow)$. Fix $p \in X$, $\varepsilon > 0$.

Let B = ε ball about $f(p)$

Then $p \in f^{-1}(B)$, which is open by assumption.

Since p is interior pt of $f^{-1}(B)$, $\exists$ some ball $N_\delta(p) \in f^{-1}(B)$.

This δ is what we want, since $f(N_\delta(p)) \subset B$.

So f is conti #.

Consequences : Thm $X \xrightarrow{f} Y \xrightarrow{g} Z$. f, g conti.

$\Rightarrow g \circ f$ is conti.

pf: Given $\mathcal{U}$ open in Z.

$\Rightarrow g^{-1}(\mathcal{U})$ is open in Y.

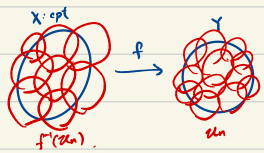
$\Rightarrow f^{-1}(g^{-1}(\mathcal{U}))$ is open in X

$\Rightarrow (g \circ f)^{-1}(\mathcal{U})$ is open in X #.

Thm : $f: X \to Y$ conti $\Leftrightarrow$ $\forall$ closed K in Y, $f^{-1}(K)$ is closed in X.

proof idea: $f^{-1}(K) = [f^{-1}(K^c)]^c$.

Thm : $f: X \to Y$ conti, X cpt $\Rightarrow$ $f(X)$ cpt.



(proof) ① Let $\{V_\alpha\}$ cover of $f(X)$.

② Let $\{\mathcal{U}_\alpha\} = \{f^{-1}(V_\alpha)\}$.

③ By cpt of X, $\exists$ finite subcover : $\mathcal{U}_{\alpha_1}, \cdots, \mathcal{U}_{\alpha_n}$

④ Then $V_{\alpha_1}, \cdots, V_{\alpha_n}$ cover $f(X)$. #.

Cor. $f: X_{\text{cpt}} \to \mathbb{R}$, then $f(X)$ closed and bounded.

$f: X_{\text{cpt}} \to \mathbb{R}$, then f achieves its maximum and minimum.

Thm : If $f: X \to Y$ bijection, conti, X: cpt, $\Rightarrow$ f^{-1} is conti.

pf : $\mathcal{U}$ open in X $\Rightarrow$ $\mathcal{U}^c$ is closed $\Rightarrow$ $\mathcal{U}^c$ is cpt $\Rightarrow$ $f^{-1}(\mathcal{U}^c)$ is cpt. $\Rightarrow$ $f^{-1}(\mathcal{U}^c)$ is closed, $\Rightarrow$ $f^{-1}(\mathcal{U})$ is open #