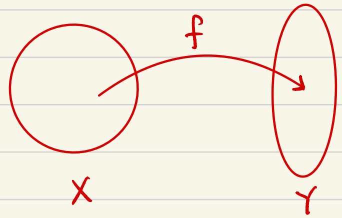
20: Functions - Limits and Continuity

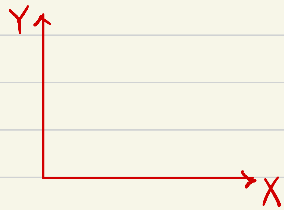
FUNCTIONS.

Let X and Y be metric spaces, $f: X \to Y$

- Visualize : f as mapping.



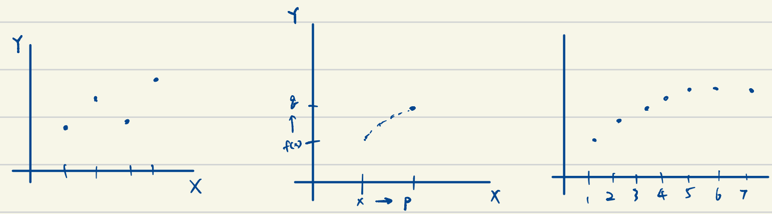
- f as graph



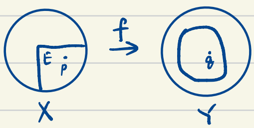
- Recall : Know what this means $\lim_{n\to\infty} x_n = x$

What is this ? $\lim_{x\to p} f(x) = q$

Ex



Def : X, Y metric, $E \subset X$, $p \in$ lim pts of E. let $f: E \to Y$



To say " $f(x) \to q$ as $x \to p$ " or $\lim_{x\to p} f(x) = q$.

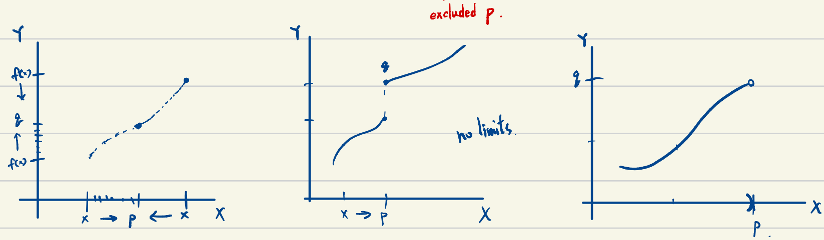
if $\exists\, q \in Y$ s.t.

$\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t.

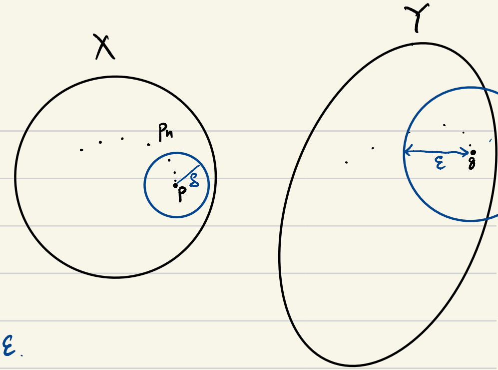
$\forall x \in E$, $0 < d(x, p) < \delta \Rightarrow d(f(x), q) < \varepsilon$.

excluded p.

Ex:



To show convergence, Given $\varepsilon > 0$, find a δ that match.



Thm $\lim_{x\to p} f(x) = q$ iff $\forall$ seq $\{p_n\}$ in E
$p_n \neq p$. $p_n \to p$.
we have $f(p_n) \to q$.
(seq. conv.).

<u>proof</u> ($\Rightarrow$) Given $\varepsilon > 0$

$\exists\, \delta > 0$ s.t. $0 < d(x,p) < \delta \Rightarrow d(f(x), q) < \varepsilon$.

So for a given $\{p_n\}$ as above,

$\exists\, N$ s.t. $d(p_n, p) < \delta$,

so $n \geq N$ implies $d(f(p_n), q) < \varepsilon$ #.

($\Leftarrow$). If $\lim_{x\to p} f(x) \neq q$, then $\exists\, \varepsilon > 0$ s.t. $\forall \delta > 0$ s.t.
$\exists x \in E$ s.t. $0 < d(x,p) < \delta$, but $d(f(x), q) > \varepsilon$ — (*).

- We propose <u>bad</u> seq.

use $\delta_n = \frac{1}{n}$, choose x_n by (*).

Then $x_n \to p$, but $d(f(x), q) \geq \varepsilon$ by (*),

so $f(x_n) \not\to q$.

From thms on seqs
- limits are unique,
- limits of sums are sums of limits.

$\lim f(x)g(x) = \lim f(x) \lim g(x)$.

CONTINUOUS FUNCTIONS.

Def: X, Y metric space. $p \in E \subset X$, $f: E \to Y$.
Say f is continuous at p, p is in E.
if $\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t.
$\forall x \in E$, $d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon$.
x may be p.

Thm: If p is a lim pts of E.

f contin at p $\Leftrightarrow$ $\lim_{x\to p} f(x) = f(p)$.

Also, if x_n is convergent seq.

f contin $\Leftrightarrow$ $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$.

Cor Sums, prods. of contin. funcs are conti.

quotent f/g. $(g\neq 0)$

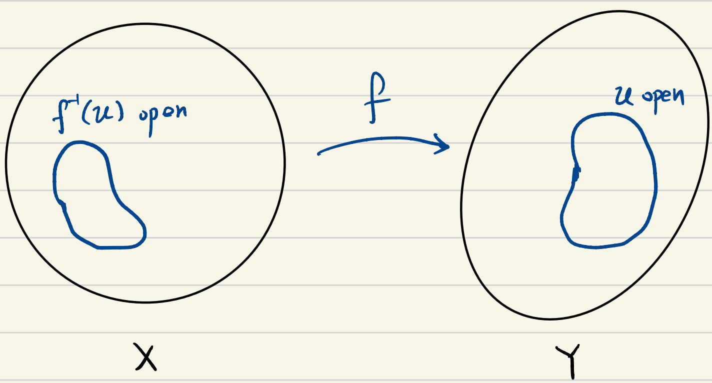
Cor f, g $X \to \mathbb{R}^k$, s.t. $f = (f_1, f_2, \cdots, f_k)$

a) f contin $\Leftrightarrow$ each f_i conti.

b) $f+g$, $f\cdot g$ conti.

(pf). a) $|f_i(x) - f_i(y)| \leq \| f(x) - f(y) \|$
idea
b) use components in part (a).

Thm $f: X \to Y$ conti $\Leftrightarrow$ $\forall$ open sets $\mathcal{U}$ in Y
$f^{-1}(\mathcal{U})$ is open in X.



prove next time.