19. Series Convergence Tests, Absolute Convergence

last time:

$\sum \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

Cauchy Criterion

Comparison test.

Thm (Root test). Given $\sum a_n$

let $\alpha = \limsup \sqrt[n]{|a_n|}$

Then if $\alpha < 1 \Rightarrow$ series converges.

if $\alpha > 1 \Rightarrow$ series diverges.

if $\alpha = 1 \Rightarrow$ test inconclusive

(pf): By comparison with geometric series.

- If $\alpha < 1$, choose β s.t. $\alpha < \beta < 1$

  Then $\exists N$ s.t. $n \geq N \Rightarrow \sqrt[n]{|a_n|} < \beta$ (by def of limsup).

  So $|a_n| < \beta^n$ for $n \geq N$.

  But $\sum \beta^n$ converges, so $\sum |a_n|$ converges. $\Rightarrow \sum a_n$ converges.

- If $\alpha > 1$, $\exists$ subseq. $\sqrt[n]{|a_{n_k}|} \longrightarrow \alpha > 1$

  so $|a_{n_k}| > 1$ for infinite many terms,

  so the terms $\not\longrightarrow 0$, so the series diverges.

- If $\alpha = 1$, note $\sum \frac{1}{n}$ div, $\sum \frac{1}{n^2}$ conv, but $\alpha = 1$ #.

Thm (Ratio test). $\sum a_n$ converges if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$

diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for n large enough.

For the series:

$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \dots$

$1 \ \frac{1}{2} \ 1 \ \frac{1}{2} \ 1 \ \frac{1}{2} \ 1.$

Ratio test tells us nothing.

(pf) (comparison): We have $\left|\frac{a_{n+1}}{a_n}\right| < \beta < 1$ for $n > N$ some N.

$$|a_{n+1}| < \beta |a_n| < \beta^2 |a_{n-1}| < \beta^3 |a_{n-2}|$$

$$|a_{N+k}| < \cdots < \beta^k |a_N|.$$

$$\sum_{k=0}^{\infty} a_{N+k} \leq a^N \sum \beta^k \text{ converges. } \#.$$

For div, see terms $\not\to 0$ #.

POWER SERIES.

If $C_n \in \mathbb{C}$ complex, $\sum_{n=0}^{\infty} C_n z^n = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + \cdots$ is a power series in z a complex variable.

For what z is this series converges?

Thm. If $\alpha = \limsup \sqrt[n]{|C_n|}$, let $R = \frac{1}{\alpha}$ (radius of convergence)

then $\sum C_n z^n$ conv if $|z| < R$

div. if $|z| > R$

(pf. idea) $\sqrt[n]{|a_n|} = |z| \cdot \sqrt[n]{|C_n|}$

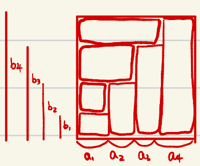
- Two seqs $\{a_n\}$, $\{b_n\}$, say something about $\sum a_n b_n$?

SUMMATION BY PARTS.

let $A_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$, set $A_{-1} = 0$

Then $\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_p b_p$

pf (idea)



the algebraic proof just check both sides.

Thm If A_n bounded, and $b_n \to 0$, then $\sum a_n b_n$ converges.

(pf idea). Say $|A_n| \le M$, Given some $\varepsilon > 0$, $\exists N$ s.t. $b_N \le \frac{\varepsilon}{2M}$.

For $q \ge p \ge N$.

$$\left|\sum_{n=p}^{q} a_n b_n\right| = \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_p b_p\right|$$

$$\le M \left| \sum (b_n - b_{n+1}) + b_p + b_q \right|$$

$$\le 2M b_p \le 2M b_N \le \varepsilon \quad \#.$$

Cor $|c_1| \ge |c_2| \ge \cdots$

c_i alternative sign $\to 0$

$\Rightarrow \sum c_n$ conv.

pf: $a_n = (-1)^{n+1}$, $b_n = |c_n|$ #.

· SUMS OF SERIES. $\sum a_n + \sum b_n = \sum a_n b_n$.

PRODUCTS?

motivation: power series.

$$(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots)$$

$$= (a_0 b_0) + (a_1 b_0 + a_0 b_1) z + (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2 + \cdots$$

$c_n = \sum_{k=0}^{n} a_k b_{n-k}$. product series $= \sum c_n z^n$.

problem: $\sum c_n$ may not conv. even if $\sum a_n$, $\sum b_n$ conv.

But. **Thm** if $\underbrace{\sum a_n}_{A}$, $\underbrace{\sum b_n}_{B}$ conv. absolutely, then $\sum c_n$ converges. to AB.

ABSOLUTE CONVERGE

Def: $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Ex: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ conv, but not absolutely.

but **Thm**: if $\sum a_n$ conv. absolutely, then converges.

(pf): $\left|\sum_{k=0}^{m} a_k\right| \le \sum_{k=0}^{m} |a_k|$ #.

REARRANGEMENTS.

Q Say $\sum a_n = A$, if I rearrange the terms must it converges? **No.**

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$

Riemann: If $\sum a_n$ conv (not abs), then a rearrangement can have any limsup, liminf, you like.

If conv abs, then all arrangements have the same limits.