18 series

SERIES

Q What does this mean?

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots = \frac{\pi^2}{6}$$

Or this?

$$1-1+1-1+\cdots = \frac{1}{2} \ (?)$$

$$(1-1)+(1-1)+\cdots = 0$$

$$1+(-1+1)+(-1+1)+\cdots = 1$$

Some we learned:

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots = \frac{1}{1-1/3}$$

as a special case of

$$1+x+x^2+x^3+\cdots = \frac{1}{1-x} \quad \text{why?}$$

$$\left.\begin{aligned} 1+2+4+8+\cdots &= \frac{1}{1-2} = -1 \\ 1-2+4-8+\cdots &= \frac{1}{1-(-2)} = \frac{1}{3} \\ 1-1+1-1+\cdots &= \frac{1}{1-(-1)} = \frac{1}{2} \end{aligned}\right\} \text{Euler.}$$

- We'll define series

Given $\{a_n\}$, defined $\sum_{n=p}^{q} a_n = a_p + \cdots + a_q$

let $S_n = \sum_{k=1}^{n} a_k$ the n-th partial sum.

- Then $\{S_n\}$ is a sequence, sometimes written $\sum_{k=1}^{\infty} a_k$ called an infinite series.

may not converge, but if it does, say to S

write $\sum_{k=1}^{\infty} a_k = S$. Notes: $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$.

Q: When does a series converges?
A: When seq of partial sum does.

<u>Ex</u> $a_n = \frac{1}{n}$, Does $1+\frac{1}{2}+\frac{1}{3}+\cdots$ converge? (harmonic series).
$(S_n = 1+\frac{1}{2}+\cdots+\frac{1}{n})$.

Is $\{S_n\}$ Cauchy?
$n>m$. $|S_n - S_m| = a_{m+1} + a_{m+2} + \cdots + a_n$.

showed $S_{2m} - S_m > \frac{1}{2}$
$\Rightarrow \{S_n\}$ is not Cauchy. not convergent.

The Cauchy Criterion for series.

<u>Thm</u>: $\sum a_n$ converges $\Leftrightarrow$ $\forall \varepsilon>0$ $\exists N$ s.t. $m,n>N \Rightarrow \left|\sum_{k=n}^{m} a_k\right| < \varepsilon$.

Let $m>n$. get

<u>Cor</u>: $\sum a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$ (terms $\to 0$)
Converse is not true (harmonic series).

<u>Thm</u> (non-neg series)
If $a_n \geq 0$, then $\sum a_n$ converges $\Leftrightarrow$ partial sums are bounded.

(pf). If $a_n \geq 0$, partial sums mon. increasing
but bounded. mono. seq. converge #.

<u>Thm</u> (comparison test).
(a). If $|a_n| \leq c_n$ for (N_2) n large enough. and $\sum c_n$ converge.
then $\sum a_n$ conv.
(b). If $a_n \geq d_n \geq 0$ for n large enough,
If $\sum d_n$ diverges, then $\sum a_n$ diverges.

(pf). (a). Since $\sum c_n$ converges, $\forall \varepsilon>0$, $\exists N$ s.t. $n,m>N_1 \Rightarrow \left|\sum_{k=n}^{m} c_k\right| < \varepsilon$.
Let $N = \max\{N_1, N_2\}$. For $n,m>N \Rightarrow \left|\sum_{k=n}^{m} a_k\right| \leq \left|\sum_{k=n}^{m} c_k\right| < \varepsilon$
(b) Use (a), if $\sum a_n$ converges, so does $\sum b_n$. #.

• What to compare to?

Geometric Series.

<u>Thm</u>: If $|x|<1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

If $|x|\geq 1$, then $\sum x^n$ diverges.

(pf). If $|x|<1$. Let $S_n = 1 + x + \cdots + x^n$ Then $S_n = \frac{1-x^{n+1}}{1-x}$

So $\lim S_n = \frac{1}{1-x}\cdot \underbrace{\lim(1-x^{n+1})}_{\to 1-0} = \frac{1}{1-x}$

If $|x|\geq 1$, terms $\not\to 0$, so series $\sum x^n$ diverges.

Q $\sum \frac{1}{n^p}$ converge or diverge? For which p?

• <u>Thm</u> (Cauchy). If $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$

Then $\sum a_n$ converges $\Leftrightarrow$ $\sum 2^k a_{2^k}$ converges.

$a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$

(pf idea). compare. $S_n = a_1 + a_2 + \cdots + a_n$ $= a_1 + (a_2+a_3) + (a_4+a_5+a_6+a_7) + \cdots$

$T_k = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_k = a_1 + (a_2+a_2) + (a_4+a_4+a_4+a_4) + \cdots$

If $n<2^k$, then $S_n \leq t_k$, ("$\Leftarrow$" done).

$2S_n = 2a_1 + 2a_2 + 2(a_3+a_4) + 2(a_5+\cdots+a_8) + \cdots$

$t_k = a_1 + 2a_2 \quad 4a_4 + \quad 8a_8 \quad + \cdots$

If $n>2^k$, then $t_k \leq 2S_n$ ("$\Rightarrow$" done). #.

Application

• $\sum \frac{1}{n^p}$ conv if $p>1$, div if $p\leq 1$.

(pf). If $p\leq 0$. terms $\not\to 0$, $\sum \frac{1}{n^p}$ div.

If $p>0$, use $\sum 2^k \frac{1}{2^{kp}} = \sum 2^{k(1-p)}$ which converges $\Leftrightarrow$ $|2^{1-p}|<1$ $\Leftrightarrow$ $1-p<0$ #

$(p>1)$

• $\sum \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots$ converges. (to e)

why? It's p sums are bounded by 3.

$(1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 3)$

Conv. rapid! $e - S_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots$

$< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right)$ $\boxed{\frac{1}{n!\,n}}$

- See e is irrational.

  If $e = \frac{m}{n}$, then $0 < \underbrace{n!\,(e - s_n)}_{\text{integer}} < \frac{1}{n}$

  but there's no integer.