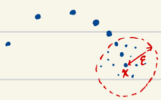
17 Complete Spaces

Thm: Compact metric spaces are complete.

(pf). Let $\{x_n\}$ be Cauchy seq. in X.



Since X cpt, it is sequentially cpt.

so $\exists$ subseq. $\{x_{n_k}\}$ converging to pt x in X

Fix $\varepsilon > 0$, $\{x_n\}$ Cauchy seq. implies $\exists N_1$ s.t. $i, j > N_1 \Rightarrow d(x_i, x_j) < \varepsilon/2$.

$\{x_{n_k}\}$ converges implies $\exists N_2$ s.t. $n_k > N_2 \Rightarrow d(x_{n_k}, x) < \varepsilon/2$.

Let $N = \max(N_1, N_2)$,

If $n > N$, then $d(x_n, x) \le d(x_n, x_{n_k}) + d(x_{n_k}, x)$ for any $n_k > N$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, given $\varepsilon > 0$, fixed N that shows $x_n \to x$ #.

Since $\{x_n\}$ was arbitrary, X is complete.

Cor: $[0, 1]$ is complete,

K cells $\subset \mathbb{R}^n$ complete.

Closed subset of a cpt space is complete.

Cor: $\mathbb{R}^n$ is complete.

(pf). If $\{x_n\}$ Cauchy, it is bounded.

(why? Fixed $\varepsilon > 0$, $\exists N$ s.t. $n, m \ge N$ implies $d(x_n, x_m) < \varepsilon$,

Let $R = \max\{d(x_N, x_1), \dots, d(x_N, x_{N-1}), \varepsilon\}$.

Seq is bounded by $B_R(x_N)$.)

So $B_R(x) \subset$ some closed ball in $\mathbb{R}^n$,

and since closed ball in $\mathbb{R}^n$ is compact $\Rightarrow$ complete.

so $\{x_n\}$ converges.

so $\mathbb{R}^n$ is complete. #.

Ex: Does $X_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ converges?

Consider $|X_n - X_m| \underset{n>m}{=} \left|\frac{1}{m+1} + \cdots + \frac{1}{n}\right| \geq \left|\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right| = \frac{n-m}{n} = 1 - \frac{m}{n}$.

Let $n = 2m$, $|X_{2m} - X_m| > \frac{1}{2}$, which implies seq not Cauchy.

so doesn't converge.

Ex: $X_1 = 1$, $X_2 = 2$, $X_n = \frac{1}{2}(X_{n-1} + X_{n-2})$ is Cauchy so converges.

Q: If X is not complete, can it be embedded to one that is?

ex: $\mathbb{Q}$ ex: $\mathbb{R}$.

Thm: Every metric space (X, d) has a completion (X^*, d).

Idea: Given X, let $X^* = \{$ all Cauchy seqs in X under equivalent relationship. $\}$

(where $\{p_n\} \sim \{q_n\}$ if $\lim_{n\to\infty} d(p_n, q_n) = 0$)

For $P, Q \subset X^*$

Let $\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n)$ where $\{p_n\}, \{q_n\}$ represent P, Q.

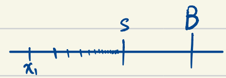
Then X^* is complete with X isometrically embedded in X^*.

BOUNDED SEQUENCES

Def: monotonely increasing seq : $s_n \le s_{n+1}$

decreasing seq : $s_n \ge s_{n+1}$

Thm: Bounded monotonely seq converges (to their sup or inf).



<pf>. Given $\{s_n\}$, let $s = \sup(\{s_n\})$.

So $\forall \varepsilon > 0$, $\exists N$ s.t. $s - \varepsilon < s_N \le s$

but then $\forall n \ge N$ $s_N < s_n < s$, so this N works for ε #.

- Write $s_n \to +\infty$ if $\forall M \in \mathbb{R}$, $\exists N$ s.t. $n > N \Rightarrow s_n > M$

  Similarly, $s_n \to -\infty$ if $\forall M$ 〃 $s_n < M$.

- Given $\{s_n\}$, let $E = \{$ subseq of limits $\}$ (allow $+\infty$, $-\infty$).

  Let $s^* = \sup E \longleftarrow \limsup s_n$, "upper limit" of E

  $s_* = \inf E \longleftarrow \liminf s_n$, "lower limit" of E

  Meanwhile, $\limsup s_n = \lim_{k\to\infty} \left(\sup_{k \ge N} s_N \right)$

  $\liminf s_n = \lim_{k\to\infty} \left(\inf_{k \ge N} s_N \right)$

Ex: If $s_k \to s$ then $\liminf s_k = \limsup s_k = s$

- $s_k = \{0.1, 3/2, 0.11, 4/3, 0.111, 5/4, 0.1111, 6/5, \cdots\}$

  $s^* = 1$, $s_* = 1/9$