16 Subsequences, Cauchy Sequences

Important Idea : To show convergence, you must find an N.

Warm-up : Consider seqs : $\{s_n\}$ 、 $\{t_n\} \in \mathbb{C}$ & $s_n \to s$ 、 $t_n \to t$

Thm : $\{s_n + t_n\} \to s+t$.

(pf) (idea : $|(s_n+t_n) - (s+t)| = |(s_n - s) + (t_n - t)| \le |s_n - s| + |t_n - t|$)

Given $\varepsilon > 0$, $\exists N_1, N_2$ s.t. $n > N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$

$n > N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{2}$

Let $N = \max\{N_1, N_2\}$,

Then for $n > N$, $|(s_n+t_n) - (s+t)| \le |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ #.

Thm : $\lim\limits_{n\to\infty} c\cdot s_n \to c\cdot s$

(idea : $|cs_n - cs| = |c|\,|s_n - s|$.

(pf). Fix $\varepsilon > 0$. Then $\exists N$ s.t. $n > N \Rightarrow |s_n - s| < \frac{\varepsilon}{|c|}$.

Then, for this N, $n > N \Rightarrow |cs_n - cs| = |c|\,|s_n - s| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon$ #

Thm : $\lim\limits_{n\to\infty} s_n t_n = st$.

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(Idea $|s_n t_n - st| = |(s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)|$)

(pf) : Given $\varepsilon > 0$, let $K = \max\{s, t, 1\}$

$\exists N_1, N_2$ s.t. $n > N_1 \Rightarrow |s_n - s| < \varepsilon/3K$

$n > N_2 \Rightarrow |t_n - t| < \varepsilon/3K$.

Let $N = \max\{N_1, N_2\}$.

Then $|s_n t_n - st| < \varepsilon^2/9K^2 + \varepsilon/3 + \varepsilon/3$

$\varepsilon<1$ $< \frac{\varepsilon}{9K} + \varepsilon/3 + \varepsilon/3 < \varepsilon$ #

SUBSEQUENCES.

$\{p_n\}$ a seq. Let $n_1 < n_2 < n_3 < \cdots$ in $\mathbb{N}$

Then $\{p_{n_i}\}$ is a subsequence.

Ex: $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\}$

Q: If $p_n \to p$, must every subsequence converge to p?

A: YES, b/c every nbd of p contain all but finitely many of points.

Ex: $\{\underline{\frac{1}{2}}, \pi, \underline{\frac{2}{3}}, \pi, \underline{\frac{3}{4}}, \pi, \underline{\ }\ \cdots \underline{\ }\}$. does not converge.

but a subseq. does. ("to subsequential limit")

Q: Must every sequence contain a conv. subseq.?

A: No. $\{1, 2, 3, \cdots\}$.

Q: If seq bounded, must have conv. subseq.?

A: No. In $\mathbb{Q}$ $\{3, 3.1, 3.14, 3.141, 3.1415, \cdots\}$.

Def: A metric space X is sequentially compact if every sequence has a convergent subsequence. (conv. to point in X).

Thm: If X is compact, then X is seg. cpt. (Fact: seg. cpt $\Rightarrow$ cpt.)

(pf): Let R = range $\{p_n\}$.

- If R is finite, then some p in $\{p_n\}$ is achieved infinitely times. Use this subseq.
- If R is infinite, then by previous thm.

  since X is cpt, R has limit point called p.

  Then use this as subseq.

CAUCHY SEQUENCES.

Q: How to tell $\{p_n\}$ converges if I don't know its limit?

Idea: If they do converge, then p_n must be getting close to each other.

Def: $\{p_n\}$ is Cauchy sequence means
$\forall \varepsilon > 0$, $\exists N$ s.t.
$m, n \geq N$ implies $d(p_n, p_m) < \varepsilon$.

Thm: $\{p_n\}$ converges $\Rightarrow$ $\{p_n\}$ is a Cauchy sequence.

(pf) Given $\varepsilon > 0$, $\exists N$ s.t. $n > N \Rightarrow d(p, p_n) < \frac{\varepsilon}{2}$

So for $n, m > N \Rightarrow d(p_n, p_m) \leq d(p, p_n) + d(p, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

The inverse is not true, $\{\mathbb{Q}\}$ is Cauchy but not conv.

Def: A metric space X is complete
if every Cauchy sequence converge to point of X.