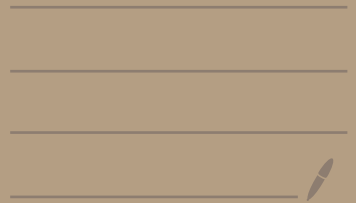


# 16 Subsequences, Cauchy Sequences

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**Important Idea:** To show convergence, you must find an  $N$ .

Warm-up: Consider seqs:  $\{s_n\}, \{t_n\} \in \mathbb{C}$  &  $s_n \rightarrow s, t_n \rightarrow t$

Thm:  $\{s_n + t_n\} \rightarrow s + t$ .

(pf) (idea:  $|(s_n + t_n) - (s + t)| = |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t|$ )

$$\text{Given } \varepsilon > 0, \exists N_1, N_2 \text{ s.t. } n > N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$$

$$n > N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{2}$$

$$\text{Let } N = \max\{N_1, N_2\},$$

$$\text{Then for } n > N, |(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \#$$

Thm:  $\lim_{n \rightarrow \infty} c \cdot s_n \rightarrow c \cdot s$

(idea:  $|c s_n - c s| = |c| |s_n - s|$ )

$$\text{(pf). Fix } \varepsilon > 0. \text{ Then } \exists N \text{ s.t. } n > N \Rightarrow |s_n - s| < \frac{\varepsilon}{|c|}$$

$$\text{Then, for this } N, n > N \Rightarrow |c s_n - c s| = |c| |s_n - s| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon \#$$

Thm:  $\lim_{n \rightarrow \infty} s_n t_n = st$  適宜

(Idea  $|s_n t_n - st| = |(s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)|$ )

(pf): Given  $\varepsilon > 0$ , let  $K = \max\{|s|, |t|\}$

$$\exists N_1, N_2 \text{ s.t. } n > N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{3K}$$

$$n > N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{3K}$$

$$\text{Let } N = \max\{N_1, N_2\}$$

$$\text{Then } |s_n t_n - st| < \frac{\varepsilon}{3K} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$\text{ex. } \rightarrow < \frac{\varepsilon}{3K} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon \#$$

## SUBSEQUENCES.

$\{P_n\}$  a seq. Let  $n_1 < n_2 < n_3 < \dots$  in  $\mathbb{N}$

Then  $\{P_{n_i}\}$  is a subsequence.

Ex:  $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$

Q: If  $P_n \rightarrow P$ , must every subsequence converge to  $P$ ?

A: YES, b/c every nbd of  $P$  contain all but finitely many of points.

Ex:  $\{\frac{1}{2}, \pi, \frac{2}{3}, \pi, \frac{3}{4}, \pi, \dots\}$  does not converge.

but a subseq. does. ("to subsequential limit")

Q: Must every sequence contain a conv. subseq.?

A: No.  $\{1, 2, 3, \dots\}$ .

Q: If seq bounded, must have conv. subseq.?

A: No. In  $\mathbb{Q}$   $\{3, 3.1, 3.14, 3.141, 3.1415, \dots\}$ .

Def: A metric space  $X$  is sequentially compact if every sequence has a convergent subsequence. (conv. to point in  $X$ ).

Thm: If  $X$  is compact, then  $X$  is seq. cpt. (Fact: seq. cpt  $\Rightarrow$  cpt.)

(pf): Let  $R = \text{range } \{P_n\}$ .

• If  $R$  is finite, then some  $p$  in  $\{P_n\}$  is achieved infinitely times. Use this subseq.

• If  $R$  is infinite, then by previous thm.

since  $X$  is cpt,  $R$  has limit point called  $P$ .

Then use this as subseq.

# CAUCHY SEQUENCES.

Q: How to tell  $\{p_n\}$  converges if I don't know its limit?

Idea: If they do converge, then  $p_n$  must be getting close to each other.

Def:  $\{p_n\}$  is Cauchy sequence means

$\forall \epsilon > 0, \exists N$  s.t.

$m, n \geq N$  implies  $d(p_n, p_m) < \epsilon$ .

Thm:  $\{p_n\}$  converges  $\Rightarrow \{p_n\}$  is a Cauchy sequence.

opf) Given  $\epsilon > 0, \exists N$  s.t.  $n > N \Rightarrow d(p, p_n) < \frac{\epsilon}{2}$

So for  $n, m > N \Rightarrow d(p_n, p_m) \leq d(p, p_n) + d(p, p_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ .

The inverse is not true,  $\{\mathbb{Q}\}$  is Cauchy but not conv.

Def: A metric space  $X$  is complete

if every Cauchy sequence converge to point of  $X$ .