# Convergence of sequences

(Last).

**Finite Intersection Condition** (FIC): A collection of sets has <u>FIC</u> if any finite subcollection has non-empty intersection.

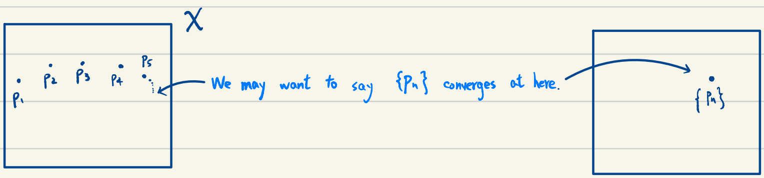
**Thm:** $K$ is compact iff every collection of closed set $\{K_\alpha\}$ that has FIC has non-empty intersection.

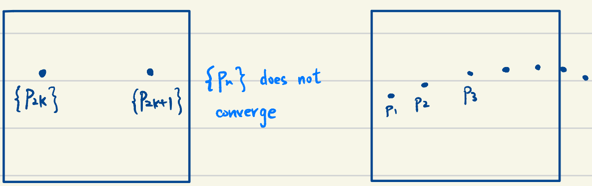
# SEQUENCES.

<u>Recall</u>: A sequence $\{p_n\}$ in $X$ is a function $f: \mathbb{N} \to X$

maps $n \mapsto p_n$, a point in $X$.

<u>Ex</u>:




<u>Def</u>: $\{p_n\}$ converges in $X$

if $\exists p \in X$ s.t. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N$ $d(p_n, p) < \varepsilon$.

Write $p_n \to p$ or $\lim_{n \to \infty} p_n = p$ Say: $p_n$ converges to $p$ or $p$ is the limit of a sequence $p_n$.

**Example**: $p_n = \frac{n+1}{n}$ in $\mathbb{R}$.

Claim $p_n \to 1$.

(The challenge of showing $p_n \to p$ is finding an $N$ for each $\varepsilon$)

(pf): Note that (i) $d(p_n, 1) = \left|\frac{n+1}{n} - 1\right| = \frac{1}{n}$.

(ii) For every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$.

Therefore, given $\varepsilon > 0$, choose $N = \lceil \frac{1}{\varepsilon} \rceil$.

Then for $n \geq N$, $d(p_n, 1) = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ #

TRUE or False ?

(A) $p_n \to p$ & $p_n \to p' \Rightarrow p = p'$

(B) $\{p_n\}$ bounded $\Rightarrow$ $p_n$ converges.

(C) $\{p_n\}$ converges $\Rightarrow$ $\{p_n\}$ bounded

(D) $p_n \to p$ $\Rightarrow$ $p$ is limit point of range of $\{p_n\}$.

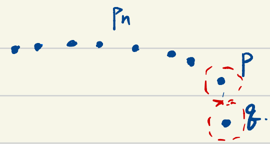
(E) $p$ is limit point of $E \subset X$ $\Rightarrow$ $\exists$ seq $\{p_n\}$ in $E$ s.t. $p_n \to p$

(F) $p_n \to p$ $\Rightarrow$ Every neighborhood of $p$ contains all but finitely many $p_n$.

(This means that $\forall N_\varepsilon(p)$, only finitely many $p_n$ are not contained in $N_\varepsilon(p)$.)

Think about it. Answer is in the following page.

T (A) $p_n \to p$ & $p_n \to p'$ $\Rightarrow$ $p = p'$




Assume $p_n \to p$, $p_n \to q$, $d(p,q) = \varepsilon > 0$ $(p \neq q)$.

Then $\exists N_p$ s.t. $n \geq N_p$ implies $d(p_n, p) < \frac{\varepsilon}{2}$

Also $\exists N_q$ s.t. $n \geq N_q$ implies $d(p_n, q) < \frac{\varepsilon}{2}$

Let $N = \max\{N_p, N_q\}$, $n \geq N$ implies

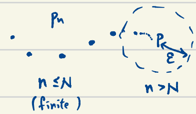
$\varepsilon = d(p,q) \leq d(p_n, p) + d(p_n, q) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ ※.

F (B) $\{p_n\}$ bounded $\Rightarrow$ $p_n$ converges.

$p_{2k} = 1$, $p_{2k+1} = 0$ for $k = 0, 1, 2, \dots$, $\{p_n\}$ bounded but not converge.

T (C) $\{p_n\}$ converges $\Rightarrow$ $\{p_n\}$ bounded

Idea :




(pf). Use $\varepsilon = 1$, then $\exists N$ s.t. $n > N \Rightarrow d(p_n, p) < 1$

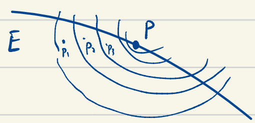
Let $R = \max\{1, d(p, p_1), \dots, d(p, p_N)\}$.

Then $\{p_n\} \subset B_R(p)$ #.

F (D) $p_n \to p$ $\Rightarrow$ $p$ is limit point of range of $\{p_n\}$.

$p_n = 1$ $\forall n$.

T (E) $p$ is limit point of $E \subset X$ $\Rightarrow$ $\exists$ seq $\{p_n\}$ in $E$ s.t. $p_n \to p$




Take point $p_1 \in N_1(p)$ in $E$, $p_2 \in N_{\frac{d(p_1,p)}{2}}(p)$ in $E$, $\dots$ $p_n \in N_{\frac{d(p_{n-1},p)}{2}}(p)$ in $E$

T (F) $p_n \to p$ $\Longleftrightarrow$ Every neighborhood of $p$ contains all but finitely many $p_n$.