14: Connected Sets, Cantor Sets

Recall: Thm: $\{K_\alpha\}$ cpt subsets of X

FIP [If any finite subcollection has non-empty intersection, then $\bigcap_\alpha K_\alpha \neq \phi$.

Cor $\{K_n\}$ a sequence cpt sets. nested, Then $\bigcap_{n\in A} K_n$ is not ϕ.

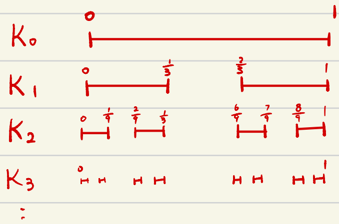
Thm: K is cpt $\Longleftrightarrow$ Any collection of closed set $\{D_\alpha\}$ satisfies the FIP:

[If every finite subcollection has non-empty intersection, then $\bigcap_\alpha D_\alpha \neq \phi$]

pf ($\Rightarrow$). Given $\{D_\alpha\}$ closed, these are closed subset of cpt X, so they are compact. Apply previous thm $_\#$

($\Leftarrow$). (practice).

• Cantor set.



Note: K_n are closed, compact, nested.

Cantor set $C = \bigcap_{n=1}^{\infty} K_n$

- C is closed (b/c intersection of closed set).
- C is perfect : it's closed and all points are limit points.
- C consists of real numbers, whose ternary expansion contains only 0 or 2.

ternary : $\sum_{k=-\infty}^{\infty} a_k 3^k$ write $\cdots a_2 a_1 a_0 . a_{-1} a_{-2} a_{-3} \cdots$

• Show C is uncountable (use diagonalization arguement).

C has non-end points of K_n point : $.020202\cdots$

Every pt is l.p.

C has measure zero : $\forall \varepsilon > 0$, C can be covered by intervals with total length less than ε.

- A base (basis) for topology is $\{V_\alpha\}$ collection such that
  $\forall x \in$ open $\mathcal{U}$, $\exists V_\alpha$ s.t. $x \in V_\alpha \subset \mathcal{U}$.
- So every open set is union of these elements.

CONNECTED SETS.

Def: Say A, B in X are separated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty.

- Say E is connected if E is not union of $\underbrace{\text{two separated sets}}_{\text{"call a separation"}}$.

Ex In $\mathbb{R}^2$, $E = \{(x,y) : x, y \in \mathbb{Q}\}$ not connected.

- E is connected $\Leftrightarrow$ E is not union of two relative open set in E
  $\Leftrightarrow$ " closed

Thm: $[a, b]$ is connected.

(pf). If not, then $\exists$ sets A, B with $a \in A$.
Let $s = \sup A$. Then $s \in \bar{A}$. So $s \notin B$
Then $s \in A$. so $s \in \bar{B}$
Then $\exists (s-\varepsilon, s+\varepsilon)$ containing no pt. of B
Then $(s-\varepsilon, s+\varepsilon) \subset A$ $\divideontimes$ $s = \sup A$ #.