13: Compactness and the Heine-Borel Theorem

FACTS (from last time)

- Compact sets are bounded and closed.
- Closed subset of compact set are compact.
- For nested closed intervals in $\mathbb{R}$, intersection is non-empty.
  (k-cells in $\mathbb{R}^k$)

- **Thm** $[a, b]$ is compact in $\mathbb{R}$.
  (k-cells) (in $\mathbb{R}^k$).

(pf) Suppose not. Then $\exists$ open cover $\{G_\alpha\}$ that has no finite subcover.
Then $\{G_\alpha\}$ covers $[a, c_1]$ and $[c_1, b]$. at least one has no finite subcover.
- WLOG, say $[a, c_1]$ has no finite subcover. (denote $I_1 = [a, c_1]$)

Then subdivide (half) using c_2, note at least one of $[a, c_2]$, $[c_2, c_1]$ has no F.S.
Continue, obtain sequence $I_1 \supset I_2 \supset I_3 \supset \dots$ nested closed intervals.
(each halved at each step, with no F.S. of $\{G_\alpha\}$)

By nested interval thm, $\exists x$ s.t. $x \in I_i \ \forall i$.
But $x \in$ some G_α of cover. So $\exists r > 0$ s.t. $N_r(x) \subset G_\alpha$.
Since I_i halved in each step, some $I_n \subset N_r(x)$, meaning single G_α covers I_n $\times$
I_i has no F.S.

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Now, we can show:

$\boxed{\begin{array}{l}\underline{\text{Heine - Boral Thm.}} \\ \text{In } \mathbb{R} \text{ (or } \mathbb{R}^n\text{), } K \text{ compact} \iff K \text{ is closed and bounded.}\end{array}}$

proof ($\Rightarrow$). already.

($\Leftarrow$). NOT TRUE IN ARBITRARY METRIC SPACE.

K bounded $\Rightarrow$ $K \subset [-r, r]$ for some $r > 0$

Since K is closed and $[-r, r]$ is compact $\Rightarrow$ K is compact #

<u>Ex</u>: Discrete metric on infinite set A.

A is closed and bounded, but not compact.

<u>EX</u>: $\mathcal{C}(\mathbb{R})$ = set of continuous bounded function $f: \mathbb{R} \to \mathbb{R}$.

$$d(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$$

• **Thm.** K is compact $\Leftrightarrow$ every infinite subset E of K has a limit point in K.

(pf) ($\Rightarrow$). If no pt of K is l.p. of E

then each $q \in E$ has nbhd V_q containing exactly one pt q of E.

$\{V_q\}$ cover E with no F.S.

($\Leftarrow$) [proof for $\mathbb{R}^k$, but true for all metric space].

sketch.

We'll show K is closed & bounded.

· Suppose K is not bounded, choose x_n s.t. $|x_n| > n$.

there has no l.p. (check).

· Suppose K is closed, $\exists\ p \notin K$ s.t. p is l.p. of K

Chousen x_n s.t. $d(x_n, p) < \frac{1}{n}$, $\{x_n\}$ has l.p. at p.

· **Cor** (Bolzano – Weierstrass Thm)

Every bounded infinite subset of $\mathbb{R}^n$ has a limit point.

pf. If subset E is bounded, then $E \subset$ compact k-cell, so has l.p. in k-cell #.

• **Thm** (Carlos, Finite Intersection Property).

$\{K_\alpha\}$ compact subsets of metric space X.

If any finite subcollection has non empty intersection,

then $\bigcap_\alpha K_\alpha \neq \phi$.

(pf) Let $U_\alpha = K_\alpha^c$ open.

Fix one K in $\{K_\alpha\}$.

If $\bigcap K_\alpha = \phi$, then $\{U_\alpha\}$ cover K compact.

$\Rightarrow \exists$ finite $\{U_{\alpha_1}, \cdots, U_{\alpha_N}\}$ cover K

so $K \cap K_{\alpha_1} \cap \cdots K_{\alpha_N} = \phi$ $\rightarrow\!\leftarrow$ #