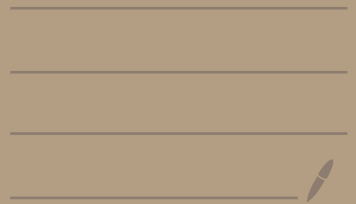


13: Compactness and the Heine-Borel Theorem



FACTS (from last time)

- Compact sets are bounded and closed.
- Closed subset of compact set are compact.
- For nested closed intervals in \mathbb{R} , intersection is non-empty.
(k -cells in \mathbb{R}^k)

• Thm $[a, b]$ is compact in \mathbb{R} .
(k -cells) (in \mathbb{R}^k).

(pf) Suppose not. Then \exists open cover $\{G_\alpha\}$ that has no finite subcover.
Then $\{G_\alpha\}$ covers $[a, c_1]$ and $[c_1, b]$. at least one has no finite subcover.
• WLOG, say $[a, c_1]$ has no finite subcover. (denote $I_1 = [a, c_1]$)
Then subdivide (half) using c_2 , note at least one of $[a, c_2]$, $[c_2, c_1]$ has no F.S.
Continue, obtain sequence $I_1 \supset I_2 \supset I_3 \supset \dots$ nested closed intervals.
(each halved at each step, with no F.S. of $\{G_\alpha\}$)

By nested interval thm, $\exists x$ s.t. $x \in I_i \forall i$.

But $x \in$ some G_α of cover. So $\exists r > 0$ s.t. $N_r(x) \subset G_\alpha$.

Since I_i halved in each step, some $I_n \subset N_r(x)$, meaning single G_α covers I_n —
 I_i has no F.S.

#

Now, we can show:

Heine - Borel Thm.

In \mathbb{R} (or \mathbb{R}^n), K compact $\iff K$ is closed and bounded.

proof (\implies) already.

(\impliedby) NOT TRUE IN ARBITRARY METRIC SPACE.

K bounded $\implies K \subset [-r, r]$ for some $r > 0$

Since K is closed and $[-r, r]$ is compact $\implies K$ is compact $\#$

Ex: Discrete metric on infinite set A .

A is closed and bounded, but not compact.

EX: $\mathcal{C}(\mathbb{R})$ = set of continuous bounded function $f: \mathbb{R} \rightarrow \mathbb{R}$.

$$d(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$$

Thm. K is compact \iff every infinite subset E of K has a limit point in K .

(pf) (\implies). If no pt of K is l.p. of E
 then each $g \in E$ has nbhd V_g containing exactly one pt g of E .
 $\{V_g\}$ cover E with no F.S.

(\impliedby) [proof for \mathbb{R}^k , but true for all metric space].

sketch. We'll show K is closed & bounded.
 • Suppose K is not bounded, choose x_n s.t. $|x_n| > n$.
 there has no l.p. (check).
 • Suppose K is closed, $\exists p \notin K$ s.t. p is l.p. of K .
 Choose x_n s.t. $d(x_n, p) < \frac{1}{n}$, $\{x_n\}$ has l.p. at p .

Cor (Bolzano-Weierstrass Thm)

Every bounded infinite subset of \mathbb{R}^n has a limit point.

pf. If subset E is bounded, then $E \subset$ compact k -cell, so has l.p. in k -cell $\#$.

Thm (Carlos, Finite Intersection Property)

$\{K_\alpha\}$ compact subsets of metric space X .

If any finite subcollection has non empty intersection,
 then $\bigcap_\alpha K_\alpha \neq \emptyset$.

(pf) Let $U_\alpha = K_\alpha^c$ open.

Fix one K in $\{K_\alpha\}$.

If $\bigcap K_\alpha = \emptyset$, then $\{U_\alpha\}$ cover K compact.

$\implies \exists$ finite $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ cover K

so $K \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n} = \emptyset \nrightarrow \#$