12: Relationship of Compact Sets to Closed Sets

Recall: A set K in metric space X is compact if every open cover of K has a finite subcover.

Thm: $E \subset Y \subset X$

E is open in $Y \Leftrightarrow E = Y \cap G$ for some G open in X.

Thm $K \subset Y \subset X$, K compact in $Y \Leftrightarrow K$ compact in X.

(pf): $(\Rightarrow)$ (Assume K cpt in Y)

START HERE → Consider open cover $\{U_\alpha\}$ of K in X

Let $V_\alpha = U_\alpha \cap Y$. Then $\{V_\alpha\}$ covers K in Y.

By cptness of K in Y, $\exists$ finite subcover $\{V_{\alpha_1}, \cdots, V_{\alpha_n}\}$

Then $\{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$ are finite subcover for K in X as desired #.

$(\Leftarrow)$. Consider open cover $\{V_\alpha\}$ of K in Y.

by thm above, $\exists U_\alpha$ s.t. $U_\alpha \cap Y = V_\alpha$.

$\{U_\alpha\}$ cover K in X, so $\exists$ finite subcover $\{U_{\alpha_i}\}_{i=1}^{N}$.

Then $\{V_{\alpha_i}\}_{i=1}^{N}$ is finite subcover of $\{V_\alpha\}$ for K in Y #.

MORAL: Compactness is an intrinsic property of a set.

Thm: Compact set are closed.

(pf). K cpt, consider $p \notin K$, We'll show p has nbhd not intersecting K.

(so p is interior to K^c).

$\forall q \in K$, let $V_q = N_{r/2}(q)$, $U_q = N_{r/2}(p)$, where $r = d(p, q)$.

Notice $\{V_q\}$ cover K, so by cptness of K, $\exists$ finite subcover $\{V_{q_1}, \cdots, V_{q_n}\}$.

Then $W = U_{q_1} \cap U_{q_2} \cap \cdots \cap U_{q_n}$ is open

Claim $W \cap V_{q_i} = \phi$ for each i. Since $W \subset U_{q_i}$ and $U_{q_i} \cap V_{q_i} = \phi$.

$\Rightarrow W$ is the desired nbhd #

EX $(0,1)$ is not compact.

EX $\mathbb{R}$ (in $\mathbb{R}$) is not compact. b/c not bounded, though it is closed.

Thm A closed subset B of cpt set K is cpt.

(pf). Let $\{U_\alpha\}$ be open cover of B.

Notice B^c is open. So $\{U_\alpha\} \cup \{B^c\}$ is open cover of K.

By cptness, $\exists$ finite subcover $\{U_{\alpha_1}, \cdots, U_{\alpha_N}, B^c\}$

Notice $B^c \cap B = \phi$, so $\{U_{\alpha_1}, \cdots, U_{\alpha_N}\}$ covers B,

and it is finite subcover as desired #.

- **Cor** F closed, K cpt in metric space X, then $F \cap K$ is compact.

Thm Nested closed intervals in $\mathbb{R}$ are not empty.

($I_n = [a_n, b_n]$

Nested: if $m > n$, then $a_n \le a_m \le b_m \le a_n$)

(pf) Let $X = \sup\{a_i\}$, exists b/c they're bounded by b_1

Clearly, $x \ge a_i$ for all i, b/c it's the sup.

$x \le b_m$ for all m, b/c b_m is an u.b. for all a_m. #.

ASIDE PROOF: $\mathbb{R}$ is uncountable.

pf) Suppose $\mathbb{R} = \{x_1, x_2, x_3, \cdots\}$ countable.

Choose I_1 missing x_1, $I_2 \subset I_1$, missing x_1, x_2, $I_3 \subset I_2$, missing $x_1, x_2, x_3, \ldots$

Nested sequence $\Rightarrow \exists\, x \in \cap I_n$, x is not in list #.