11: Compact Sets

Compact sets are the next best thing to be finite.

Definition

- An open cover of E in X is a collection of open sets $\{G_\alpha\}$ whose union "covers" (contains) E

- A subcover of $\{G_\alpha\}$ is a sub collection $\{G_{\alpha_r}\}$ that still covers E.

Ex. In $\mathbb{R}$,

$[\frac{1}{2}, 1)$ has cover $\{V_n\}_{n=3}^{\infty}$ where $V_n = (\frac{1}{n}, 1-\frac{1}{n})$.

Also $\{(0,2)\}$.

$\{W_x\}_{x\in[\frac{1}{2},1]}$ where $W_x = N_{\frac{1}{2}}(x)$

Q Given cover, do we need all the sets to still cover?

$\{V_n\}_{n=3}^{\infty}$ has subcover $\{V_n\}_{n=12}^{\cancel{\infty}}$

$\{W_x\}$ " $\{W_{10/5}, W_{10/6}, W_{10/7}, W_{10/8}, W_{10/9}\}$ ← a finite subcover.

Ex $[0,1]$ in $\mathbb{R}$

has cover by $\{V_n\} \cup \{W_0, W_1\}$

A finite subcover is $\{W_0, W_1, V_{11}\}$.

Def: Say a set K is compact (in X) if any open cover of K contains a finite subcover.

(So K not compact if $\exists$ open cover of K that has no finite subcover.)

Warning: No saying "there's a finite cover".

Ex: $[\frac{1}{2}, 1)$ is not compact (see $\{V_n\}$).

$\mathbb{Z}$ in $\mathbb{R}$ is not compact.

Ex: $[0, 1]$ may be compact but I'd need to check every open cover.

Thm: Finite sets are open.

(pf). Consider an open cover $\{G_\alpha\}$ covering $x_1, \cdots, x_N$.

$\forall x_i$, choose one G_{α_i} that contains x_i,

Then $\{G_{\alpha_i}\}_{i=1}^N$ covers the set #.

Thm: Compact set are bounded.

(Define a set K is bounded if $K \subset N_r(x)$ for some $x \in X$, $r \in \mathbb{R}$)

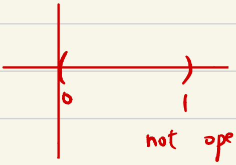
(pf) Let K be a compact set.

Let $B(x) = N_1(x)$. $\{B(x)\}_{x \in K}$ is an open cover of K.

By compactness, $\exists$ finite subcover $\{B(x_i)\}_{i=1}^N$

Let $R = \max_{i \neq j} \{d(x_i, x_j)\}$ this max exists since set $\{x_1, \cdots, x_N\}$ is finite.

Then $N_{R+2}(x_1)$ contains all K #



open in $\mathbb{R}$

not open in $\mathbb{R}^2$.

"RELATIVE" OPEN SETS.

If $Y \subset X$ metric, then Y "inherits" metric from X.

Def: A set $\mathcal{U}$ is open in Y (relative to Y) if every pt of $\mathcal{U}$ is an interior pt of $\mathcal{U}$

Thm: $E \subset Y \subset X$

E open in Y $\iff$ $E = Y \cap G$ for some G open in X.

proof idea: $(\Leftarrow)$. Use "if $N_r(x) \subset G$, then $N_r(x) \cap Y$ is nbhd of $x \in Y$, $x \in E$.

$(\Rightarrow)$ Every pt x has $N_r(x) \subset Y \cap E$.

Then $\bigcup_{x \in E} N_r(x)$ in X is open, called G #.