10: The Relationship Between Open and Closed Sets

Recall : A set E in metric space X is open if every pt is an interior pt.

A set K is closed if K contains all its limit pts.

• Def The closure of set A is $\bar{A} = A \cup \underbrace{A'}_{\text{limit pt of } A}$.

Thm $\bar{A}$ is closed set.

pf). Consider p a limit pt of $\bar{A}$

Want to show $p \in \bar{A}$.

Consider a nbhd N of p, Assume $p \notin A$.

we'll show N contains a pt of A.

Since p is l.p. of $\bar{A}$, N contains a pt q of $\bar{A}$

If $q \in A$, we found a desired point.

If $q \notin A$, then q is a l.p. of A.

Consider N' nbhd of q s.t. $N' \subset N$.

So $q' \in N$, the desired pt. # (since nbhd are open)

lemma : nbhd are open.

let $a = d(p,q) < r$.

let $r' = r - a$.

Claim $N_{r'}(q) \subset N_r(p)$.

If $d(x,q) < r'$,

$d(x,p) < d(x,q) + d(q,p) < r' + a - r$.

Thm : E closed $\Leftrightarrow$ $E = \bar{E}$

(pf). ($\Rightarrow$): $E' \subset E$ so $E \cup E' \subset E$. so $\bar{E} \subset E$

Since $E \subset \bar{E}$, $E = \bar{E}$.

($\Leftarrow$): $E = \bar{E}$ $\Rightarrow$ E contains all its l.p's.

Thm : If $E \subset$ closed set F, then $\bar{E} \subset F$.

(pf). p is lp of E $\Rightarrow$ p is lp of F.

But F contains its lp $\Rightarrow$ F contain lp's of E $\Rightarrow$ $\bar{E} \subset F$ #.

RELATIONSHIP BETWEEN OPEN & CLOSED SETS

Thm : E is open $\Leftrightarrow$ E^c is closed.

(E^c : the complement of E, $E^c = X \setminus E = \{p \in X : p \notin E\}$.)

X: metric space

pf : E open $\Leftrightarrow$ any pt $x \in E$ is an interior pt

$\Leftrightarrow$ $\forall x \in E$, $\exists$ nbhd N of x s.t. N is disjoint from E^c

$\Leftrightarrow$ $\forall x \in E$, x is not l.p. of E^c.

$\Leftrightarrow$ E^c contains all its l.p.'s. #.

Unions & Intersections.

Lemma : $\{E_\alpha\}$ collection of sets.

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$$

(pf) $x \in LHS \iff x \notin$ any E_α

$\iff x \in E_\alpha^c \quad \forall \alpha$

$\iff x \in \bigcap_\alpha E_\alpha^c$ #

Thm (a) Arbitrary union of open sets is open.

(b) Arbitrary intersection closed closed

(c) Finite intersection of open sets is open.

(d) Finite union closed closed.

pf, (a). $x \in \bigcup_\alpha \mathcal{U}_\alpha$, $\mathcal{U}_\alpha$: open. $\Rightarrow x \in$ some $\mathcal{U}_\alpha$

so x has nbhd N s.t. $N \subset \mathcal{U}_\alpha \Rightarrow N \subset \bigcup_\alpha \mathcal{U}_\alpha$. #

(b) Say B_α closed, Then $\mathcal{U}_\alpha = B_\alpha^c$ is open.

Use lemma, $\mathcal{U}_\alpha^c = B_\alpha$ $\bigcup_\alpha B_\alpha^c$ is open $\Rightarrow \left(\bigcup_\alpha B_\alpha^c\right)^c$ is closed.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; \bigcap_\alpha B_\alpha$.

(c) $\exists N_{r_i}(x)$ for each $\mathcal{U}_i$.

Let $r = \min(r_1, \cdots, r_n)$

$N_r(x) \subset \bigcap_{i=1}^{n} \mathcal{U}_i$.

- E is dense in metric space X.

if every pt of X is l.p. of E or in E.

$\iff \bar{E} = X$

$\iff$ Every open set of X contains $p \in E$.

Ex : $\mathbb{Q}$ is dense in $\mathbb{R}$.