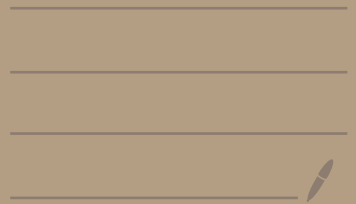


10: The Relationship Between Open and Closed Sets



Recall = A set E in metric space X
is open if every pt is an interior pt.

A set K is closed if
 K contains all its limit pts.

• Def The closure of set A is $\bar{A} = A \cup \underbrace{A'}_{\text{limit pt of } A}$.

Thm \bar{A} is closed set.

pf.) Consider p a limit pt of \bar{A}
Want to show $p \in \bar{A}$.

Consider a nbhd N of p . Assume $p \notin A$.
we'll show N contains a pt of A .

Since p is l.p. of \bar{A} , N contains a pt g of \bar{A}

If $g \in A$, we found a desired point.

If $g \notin A$, then g is a l.p. of A .

Consider N' nbhd of g s.t. $N' \subset N$.

So $g' \in N$, the desired pt. \neq (since nbhd are open)

lemma = nbhd are open.

let $a = d(p, g) < r$.

let $r' = r - a$.

Claim $N_{r'}(g) \subset N_r(p)$.

If $d(x, g) < r'$,

$$d(x, p) < d(x, g) + d(g, p) < r' + a = r.$$

Thm: E closed $\iff E = \bar{E}$

(pf). (\implies) : $E' \subset E$ so $E \cup E' \subset E$. so $\bar{E} \subset E$

Since $E \subset \bar{E}$, $E = \bar{E}$.

(\impliedby) : $E = \bar{E} \implies E$ contains all its l.p.'s.

Thm: If $E \subset$ closed set F , then $\bar{E} \subset F$.

(pf). p is l.p. of $E \implies p$ is l.p. of F .

But F contains its l.p. $\implies F$ contains l.p.'s of $E \implies \bar{E} \subset F$. #

RELATIONSHIP BETWEEN OPEN & CLOSED SETS

Thm: E is open $\iff E^c$ is closed.

(E^c : the complement of E , $E^c = \underset{\substack{\uparrow \\ \text{metric space}}}{X} \setminus E = \{p \in X : p \notin E\}$.)

pf: E open \iff any pt $x \in E$ is an interior pt

$\iff \forall x \in E, \exists$ nbhd N of x s.t.

N is disjoint from E^c

$\iff \forall x \in E, x$ is not l.p. of E^c .

$\iff E^c$ contains all its l.p.'s. #

Unions & Intersections.

Lemma: $\{E_\alpha\}$ collection of sets.

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$$

$$\begin{aligned} \text{(pf)} \quad x \in \text{LHS} &\Leftrightarrow x \notin \text{any } E_\alpha \\ &\Leftrightarrow x \in E_\alpha^c \quad \forall \alpha \\ &\Leftrightarrow x \in \bigcap_\alpha E_\alpha^c \quad \# \end{aligned}$$

Thm (a) Arbitrary union of open sets is open.

(b) Arbitrary intersection closed closed

(c) Finite intersection of open sets is open.

(d) Finite union closed closed.

pf, (a). $x \in \bigcup_\alpha U_\alpha$, U_α : open. $\Rightarrow x \in \text{some } U_\alpha$

so x has nbhd N s.t. $N \subset U_\alpha \Rightarrow N \subset \bigcup_\alpha U_\alpha$. #

(b) Say B_α closed, Then $U_\alpha = B_\alpha^c$ is open.

Use lemma, $U_\alpha^c = B_\alpha$ $\bigcap_\alpha B_\alpha^c$ is open $\Rightarrow \left(\bigcap_\alpha B_\alpha^c\right)^c$ is closed.
 $\bigcap_\alpha B_\alpha$

(c) $\exists N_{r_i}(x)$ for each U_i .

Let $r = \min(r_1, \dots, r_n)$

$N_r(x) \subset \bigcap_{i=1}^n U_i$.

• E is dense in metric space X .

if every pt of X is l.p. of E or in E .

$$\Leftrightarrow \bar{E} = X$$

\Leftrightarrow Every open set of X contains $p \in E$.

Ex: \mathbb{Q} is dense in \mathbb{R} .