09: Limit Points

Recall : (X, d) metric space.

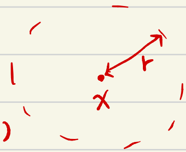
$\uparrow$ set $\quad \uparrow$ metric.

ex: $(\mathbb{R}^n$, Euclidean$)$

ex: $(X$, discrete metric$)$

$$\delta(p,q) = \begin{cases} 0 & \text{if } p=q \\ 1 & \text{if } p \neq q. \end{cases}$$

Recall : "Open ball" or "nbhd" $N_r(x)$



Ex $(X$, discrete$)$, open ball are single pts (if $r \leq 1$)

or all X (if $r > 1$)

Q : When does set E approach a point p ?

Def : A pt $p \in X$ is a limit pt of E

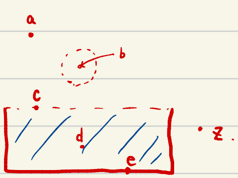
if every nbhd at p contains a point $q \in E$, $q \neq p$.

Ex In $\mathbb{R}$, $G = \{ \frac{1}{n} : n \in \mathbb{N} \}$, 0 is a limit pt.

Ex In $\mathbb{R}^2$, consider B :

b, c, d, e are limit pts.

a, z are not



So A pt p is not a limit pt of E

if $\exists$ nbhd N at p s.t. N does not contain any other pt of E.

Def p is an isolated pt of E if $p \in E$ and p is not a limit pt of E.

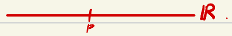
Ex: all pts of G are isolated.

Def p is an interior pt of E if $\exists$ nbhd N of p s.t. $N \subset E$.

Ex G has no interior pt. d is interior pt of B.

EX: In $\mathbb{R}$, consider ϕ, $\mathbb{R}$, $\mathbb{Q}$. limit pt ?, interior pt ?, isolated pt ?

In $(\mathbb{R}$, discrete$)$



has nbhd $N_1(p)$ that contains no other pt.

every pt in $(\mathbb{R}$, discrete$)$ is interior pt.

- $E = \mathbb{Q}$ in $\mathbb{R}$.

all pt in $\mathbb{R}$ is limit pt.

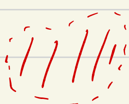
Thm If p is a limit pt of E, then every nbhd of p contains infinitely many pts of E.

pf: If not, $\{e_1, \cdots, e_n\} = N$, Let $r = \min_{i \neq j} \{d(e_i, e_j)\}$

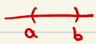
Then $N_r(p)$ has no other pt of E.

Def: A set E is open if every pt of E is an interior pt.

Ex "nose" of B is open.



Ex: In $\mathbb{R}$,



interval $(a,b) = \{x : a < x < b\}$ is open

<u>Def</u> : A set E is closed if E contains all its limit pts.

<u>EX</u> In $\mathbb{R}$, $\{p\}$ is closed.

interval $[a,b] = \{x : a \le x \le b\}$ is closed.

$\left.\begin{array}{l}(a,b] \\ [a,b)\end{array}\right\}$ "half open" interval are not both open and closed.

<u>Ex</u> : Mouth of B is not closed, Can we close it?

<u>Def</u> : Let E' a set of limit pt of E.

The closure of E is $\bar{E} = E \cup E'$

Is $\bar{E}$ closed? $\boxed{\text{Yes}}$