08: Cantor Diagonalization and Metric Spaces

TODAY: MORE ABOUT CARDINALITY
METRIC SPACE.

Thm: The union of countable sets is countable

proof: Say each $A_1, A_2, A_3, \cdots$ are countable.

Then

$$
\begin{aligned}
A_1 &= \{a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, \cdots\} \\
A_2 &= \{a_{21}, a_{22}, a_{23}, \quad \cdots \quad\} \\
A_3 &= \{a_{31}, a_{32}, a_{33}, \quad \cdots \quad\}
\end{aligned}
$$

So $\bigcup_{i=1}^{\infty} A_i$ is countable.

Notation: Use $\bigcup_{\alpha \in J} A_\alpha$ for possibly uncountable collection, J: index set.

Ex: The set of computer programs is countable.

Recall: $\mathbb{R}$ is not countable (say $\mathbb{R}$ is uncountable)

so, there are real numbers that are not computable
(we can't be specified to arbitrary precision.)

- Given set A, the power set 2^A is the set of all subsets of A.

Ex: $A = \{☺, □, \triangle\}$, then $D = \{☺, \triangle\}$, $E = \{\triangle\}$, $\phi, \cdots$

$D \mapsto 1 \quad 0 \quad 1$

$E \mapsto 0 \quad 0 \quad 1$

$\phi \mapsto 0 \quad 0 \quad 0$

are elements of 2^A which has 2^3 elements.

Cantor's Thm (diagonal argument).

For any A, we have $A \nsim 2^A$.

proof: (contradiction).

Suppose $\exists$ bijection $f: A \to 2^A$

Then $a \mapsto f(a)$ a subset of A.

Let $B = \{a : a \notin f(a)\}$

- So if $B = f(x)$ for some $x \in A$

If $x \in B$, then $x \notin f(x) = B$ $\times$

If $x \notin B$, then $x \notin f(x) \Rightarrow x \in B = f(x)$ $\times$

Ex. $2^A \sim \{f : A \to \{0,1\}\}$

$2^{\mathbb{R}} \sim$ all functions from $\mathbb{R} \to \{0,1\}$

- Cor There are infinitely many cardinalities

$0, 1, 2, 3, 4, \ldots, \aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\alpha, \ldots$

$\uparrow$

car$(\mathbb{Z})$ car$(\mathbb{R})$?

METRIC SPACES.

Q: How to measure distance? in $\mathbb{R}^n$? in sequences?

Def: A set X is a metric space if $\exists$ metric $d: X \times X \to \mathbb{R}$ such that $\forall\, p, q \in X$,

ⓐ $d(p,q) \geq 0$ ($=0$ iff $p=q$) (iff: if and only if)

ⓑ $d(p,q) = d(q,p)$.

ⓒ $d(p,q) \leq d(p,r) + d(r,q)$ $\forall\, r \in X$ ($\forall$: for all.)

Ex: $\mathbb{R}$ with $d(x,y) = |x-y|$

$\mathbb{R}^n$ with $d(\vec{x}, \vec{y}) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$

Write $(\mathbb{R}, d)$ } Euclidean metric in $\mathbb{R}^n$

Ex: $\mathbb{R}^n$ with $d(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$

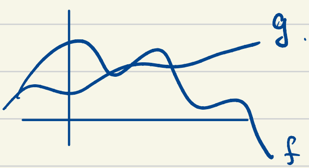
staircase metric.

Ex: space of functions.

$d(f,g) = \int_a^b |f-g|\,dx$ space of conti func. on $[a,b]$. $(\mathcal{C}([a,b]))$

or

$d(f,g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$



Open Ball $N_r(x) = \{ y : d(x,y) < r \}$. (nbhd)

Closed Ball $\overline{N_r(x)} = \{ y : d(x,y) \leq r \}$.

Def: Say $p \in X$ is a limit pt of E if every nbhd of p contains a pt $q \neq p$ s.t. $q \in E$.