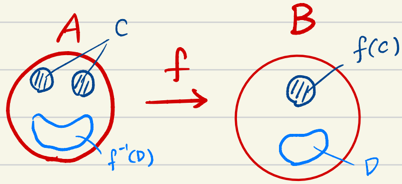
07: Countable and Uncountable Sets

TODAY : HOW TO CONT.

- Recall $f : \overset{\text{domain}}{A} \to \overset{\text{co-domain}}{B}$.

"maps" $x \mapsto f(x)$



- If $C \subset A$, $D \subset B$

define $f(C) = \{f(x) : x \in C\}$ the image of C.

$f^{-1}(D) = \{x : f(x) \in D\}$ the inverse image of D

- When $f(A) = B$, say f is onto (a surjection) $\twoheadrightarrow$

When $f(x) = f(y)$ implies $x = y$, say f is 1-1 (an injection) $\hookrightarrow$

When f is 1-1 and onto, call f a bijection $\hookrightarrow\!\!\!\!\!\rightarrow$

and say A and B are in "1-1 corresponding"

Write $A \sim B$

- Elementary Counting use $A = J_n = \{1, 2, 3, \cdots, n\}$. or $J_0 = \phi$.

Ex $\{1, ☺, ☺, \pi\} = A$ Say $|A| = 4$.

$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow \qquad \updownarrow$

$1\ \ 2\ \ 3\ \ 4 \qquad J_4$

Def : Call A finite if $A \sim J_n$, else A infinite.

Call A countable if $A \sim \mathbb{N}$.

Ex : $\mathbb{N}$ is countable : use $f : \mathbb{N} \to \mathbb{N}$ where $f(x) = x$.

Ex A sequence $x_1, x_2, \cdots$ is countable

$\updownarrow\ \updownarrow$

$1\ \ 2\ \cdots$

NOTE : A set that can be "listed" in sequence is countable !

Ex : $\{2, 3, 4, 5, \cdots\}$ is countable. use $f(n) = n+1$

$\{1, 2, 3, \cdots, k-1, k+1, k+2, \cdots\}$, use $f(n) = n$ if $n < k$

$f(n) = n+1$ if $n \geq k$.

Thm: $\mathbb{N}$ is infinite.

proof by induction on n, show $\mathbb{N} \not\sim J_n$.

base case. if $\mathbb{N} \overset{g}{\sim} \{1\}$, then consider $\mathbb{N} \setminus g(1)$ is not empty.

int step: if $\mathbb{N} \not\sim J_n$, then $\mathbb{N} \not\sim J_{n+1}$.

If there were $\mathbb{N} \xrightarrow{h} J_{n+1} = \{1, 2, \dots, n+1\}$.

then $\exists$ bijection $\mathbb{N} \setminus \{h(n+1)\} \xleftarrow{\hat{h}} J_n = \{1, \dots, n\}$

$\exists$ bijection f $\downarrow f$

$$\begin{array}{ccc} \mathbb{N} \setminus \{h(n+1)\} & \xleftarrow{\hat{h}} & J_n \\ \downarrow f & \nearrow & \\ \mathbb{N}. & & \end{array}$$

bijection $\not\longrightarrow$.

• EX $\mathbb{N} \sim 2\mathbb{N} = \{2, 4, 6, 8, \cdots\}$. use $f(n) = 2n$.

even

$\mathbb{N}$, $2\mathbb{N}$ has same cardinality ("size")

• Ex $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$ is countable.

$$\begin{array}{ccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ 7 & 5 & 3 & 1 & 2 & 4 & 6 \end{array}$$

Thm: Every inf subset E of countable set A is countable.

pf idea Let $A = \{x_1, x_2, x_3, \cdots\}$.

Let $n_1 = \inf\{i : x_i \in E\}$

$n_2 = \inf\{i : x_i \in E,\ i > n_1\}$

$\vdots$

$n_k = \inf\{i : x_i \in E,\ i > n_{k-1}\}$

Then $E = \{x_{n_1}, x_{n_2}, \cdots\}$ #.

Thm: $\mathbb{Q}$ is countable.

$$\begin{array}{ccccc} 1/1 & 2/1 & 3/1 & 4/1 & \cdots \\ 1/2 & 2/2 & 3/2 & 4/2 & \\ 1/3 & 2/3 & 3/3 & 4/3 & \cdots \\ \vdots & & & & \end{array}$$

Thm: A countable $\Rightarrow$ $A \times A$ countable.

Thm: $\mathbb{R}$ is not countable.

(pf) Suppose $\exists$ bijection.

we can show it is not bijection

$1 \longleftrightarrow 0.1\,2\,3\,4\,5\,6\,7\,8$

$2 \longleftrightarrow 0.3\,1\,4\,1\,5\,9\,2\,6$

$3 \longleftrightarrow 0.1\,4\,1\,4\,2\,1\,3\,5$

$4 \longleftrightarrow 0.7\,7\,7\,7\,7\,7\,7\,7$

$5 \longleftrightarrow 0.4\,1\,3\,2\,4\,0\,8\,9$

$0.7\,7\,7\,1\,7\cdots = x^*$,

x^* is not $f(n)$ for any n.

Thm: For any A, $A \nsim 2^A$ the power set of A