# 06: Principle of Induction

# INDUCTION

- let $\mathbb{N} = \{1, 2, 3, 4, \dots\}$, natural numbers.
- Well-ordering property of $\mathbb{N}$:
  (WOP)

  $\mathbb{N}$ is well-ordering : every non-empty subset of $\mathbb{N}$ has a least element.

**Remark** : Can take WOP to be an axiom of $\mathbb{N}$.

- Principle of Induction : (POI)

  Let $S$ be a subset of $\mathbb{N}$ such that
  
  ⓐ $1 \in S$
  
  ⓑ if $k \in S$, then $k+1 \in S$.
  
  then $S = \mathbb{N}$.

## FACT : WOP $\Longleftrightarrow$ POI

proof : WOP $\Rightarrow$ POI.

(by contradiction). Suppose $S$ exists with given properties in POI, but $S \neq \mathbb{N}$.

Then $A = \mathbb{N} \setminus S$, is non-empty, has a least element by WOP, call it $n$.

Notice $n > 1$ since $1 \in S$.

Consider $n-1$, it is not in $A$, so in $S$, By prop ⓑ, $(n-1)+1 \in S \Rightarrow n \in S$.

Therefore, POI holds. # contradict to $n \in A$.

# – PROOFS BY INDUCTION:

Let $P(n)$ be statement ordered by $n \in \mathbb{N}$.

Idea: to show $P(n)$ is true for all $n$,

we'll show (a) $P(1)$ is true

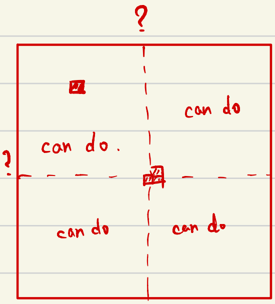
(b) if $P(n)$ is true, then $P(n+1)$ is true

Then by POI, $P(n)$ is true for all $n$.

**Strong Induction**: use (b') if $P(1), P(2), \cdots, P(k)$ are true then $P(k+1)$ is true

- **STYLE**: at start, tell reader "proof by induction on ___"
  - tell reader when you're doing "base case" & "inductive step"
    assume these are understood.
  - remind reader of conclusion at end.

• **Ex** Every $2^n \times 2^n$ chessboard with one square removed can be tiled by ⊞ .




proof (by induction on $n$ )

• For base case, see  as desired.

• For $n$ step, we can assume any $2^n \times 2^n$ board (with 1 removed) can be tiled

So consider a $2^{n+1} \times 2^{n+1}$ board with 1 removed.

Can divide board into 4 parts, one with 1 removed and 3 are $2^n \times 2^n$ boards

The first can be tiled by inductive hypt, the remaining 3 can be tiled once a tile has removed,

let's remove one in an L shape. (figu

So $2^{n+1} \times 2^{n+1}$ can be tiled. #.

**Thm** : Prove $S_n = 1 + 3 + 5 + \cdots + (2n-1)$ is a prefect square.

proof • base case $(n=1)$ holds, since $S_1 = 1 = 1^2$ as derived. claim $S_n = n^2$.

• For ind step, assume $S_n$ is square $k^2$ for some $k \in \mathbb{N}$.

We wish to show $S_{n+1}$ is perfect square, $S_{n+1} = 1 + 3 + \cdots + (2n+1)$

$= S_n + 2n + 1$