05: Complex Numbers

- EXTENDED REALS.

$\bar{R} := \mathbb{R} \cup \{-\infty, +\infty\}$

Put <u>order</u> $\forall x \in \mathbb{R}$, $-\infty < x < +\infty$.

and <u>arithmetic</u> $x + (+\infty) = +\infty$

$x + (-\infty) = -\infty$.

If $x > 0$, $x \cdot (+\infty) = +\infty$.

If $x < 0$, $x \cdot (+\infty) = -\infty$ <u>etc</u>.

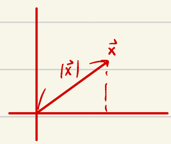
- why care? convenient, e.g. every subset in $\bar{\mathbb{R}}$ has a sup.

- Euclidean space $\mathbb{R}^k := \{(x_1, \cdots, x_k) : x_i \in \mathbb{R} \ \forall i = 1, \cdots, k\}$

Define $(x_1, \cdots, x_k) + (y_1, \cdots, y_k) = (x_1 + y_1, \cdots, x_k + y_k)$. ← addition

scalar mult. $\underset{\text{in } \mathbb{R}}{\alpha}(x_1, \cdots, x_k) = (\alpha x_1, \cdots, \alpha x_k)$

Also, $\mathbb{R}^k$ has an "inner product" $\vec{x} \cdot \vec{y} = \sum_{i=1}^{k} x_i y_i$ with <u>norm</u> length. $|\vec{x}| = (\vec{x} \cdot \vec{x})^{1/2}$



- Complex number field.

$\mathbb{R}^2$ can be given a <u>field structure</u>:

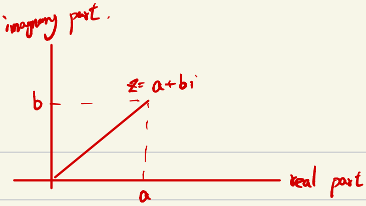
$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b) \times (c, d) = (ac - bd, ad + bc)$$

Here, the zero is $(0, 0)$, and the 1 is $(1, 0)$

Write $\mathbb{C}$, the set $\mathbb{R}^2$ with $+, \times$ as above.

- $\mathbb{C}$ extends $\mathbb{R}$: $\{(a, 0) : a \in \mathbb{R}\}$ "behaves" like $\mathbb{R}$,
- Note $\underbrace{(0, 1)}_{i} \cdot (0, 1) = (-1, 0)$



Write $a+bi$ for (a,b).

If $z=a+bi$

let $\bar{z}=a-bi$, the conjugate of z. $\cdot\,\bar{z}$

Check $\overline{z+w}=\bar{z}+\bar{w}$

$\overline{z\cdot w}=\bar{z}\cdot\bar{w}$

$z+\bar{z}=2\,\mathrm{Re}(z)$, and $z-\bar{z}=2\cdot \mathrm{Im}(z)$

$z\cdot\bar{z}=a^2+b^2$ real ≥ 0.

Define $|z|=(z\cdot\bar{z})^{1/2}$, the same as length in $\mathbb{R}^2$.

abs value.

- Suggests, in $\mathbb{C}^k=\{(z_1,\cdots,z_k): z_i\in\mathbb{C}\}$,

  the inner product $\langle\vec{x},\vec{y}\rangle:=\sum_{i=1}^{k}x_i\bar{y}_i$

property. $|z|\geq 0$, $|\bar{z}|=|z|$, $|zw|=|z|\cdot|w|$

based on
$(ac-bd)^2+(ad+bc)^2$
$=(a^2+b^2)(c^2+d^2)$.

and $|z+w|\leq|z|+|w|$ (triangle inequality).

(pf): $|z+w|^2=(z+w)\cdot(\bar{z}+\bar{w})$

$=z\cdot\bar{z}+z\cdot\bar{w}+w\cdot\bar{z}+w\cdot\bar{w}$

$=|z|^2+2\,\mathrm{Re}(z\bar{w})+|\bar{w}|^2$

$\leq|z|^2+2|z||w|+|\bar{w}|^2$

$=(|z|+|w|)^2$ this yields desired ineq.

- <u>Cauchy - Schwarz inequality.</u>

If $a_1, \cdots, a_n$ are complex numbers, then
$b_1, \cdots, b_n$

$$\left| \sum_{i=1}^{n} a_i \overline{b_i} \right|^2 \leq \sum_{i=1}^{n} |a_i|^2 \cdot \sum_{i=1}^{n} |b_i|^2$$

In $\mathbb{R}^k$, $|\vec{v} \cdot \vec{w}| \leq |\vec{v}| \cdot |\vec{w}|$

$$|\langle \vec{v}, \vec{w} \rangle|^2 \leq \langle \vec{v}, \vec{v} \rangle \cdot \langle \vec{w}, \vec{w} \rangle$$

<u>proof</u>: let $\vec{a}, \vec{b} \in \mathbb{C}^n$.

Note $0 \leq |\vec{a} - y\vec{b}|^2 = \langle a-yb, a-yb \rangle$

$$= \langle a, a \rangle - \bar{y} \langle a, b \rangle - y \langle b, a \rangle + |y|^2 \langle b, b \rangle$$

choose $y = \frac{\langle a, b \rangle}{\langle b, b \rangle}$ $\quad = \langle a, a \rangle - \frac{|\langle a, b \rangle|^2}{\langle b, b \rangle}$,

$$\Rightarrow 0 \leq \langle a, a \rangle - \frac{|\langle a, b \rangle|^2}{\langle b, b \rangle} \Rightarrow |\langle a, b \rangle|^2 \leq \langle a, a \rangle \langle b, b \rangle.$$