# 04: The Least Upper Bound Property

- Defined order
    arthmetic "+", "x"
- Also, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.
    Associate to $q \in \mathbb{Q}$, the cut $q^* = \{r \in \mathbb{Q} : r < q\}$.
    Check: $f : \mathbb{Q} \to \mathbb{R}$ preserve $+, \times, <$
    $q \mapsto q^*$
    Then $\mathbb{Q}' = \{q^* : q \in \mathbb{Q}\}$ is a subfield of $\mathbb{R}$.

<u>Notice</u>: length "$\sqrt{2}$" sets in $\mathbb{R}$
as $\gamma = \{q : q^2 < 2 \text{ or } q < 0\}$
check, use def of $\times$, that $\gamma^2 = 2^*$

- $\mathbb{R}$ has the least upper bound property!
  If $A$ is collection of cuts with u.b. $\beta$.
  let $\gamma = \cup\{\alpha : \alpha \in A\}$, a subset of $\mathbb{Q}$.
  Check: $\gamma$ is a cut & $\gamma = \sup A$

  nontriv (bdd).
  closed down, no largest number.

  $\gamma$ is an ub clearly, since $\gamma$ contains all $\alpha \in A$.
  - $\gamma$ is least upper bound since if $\delta < \gamma$, $\exists x \in \gamma \setminus \delta$
    Then $x \in$ some $\alpha \in A$, not in $\delta$, so $\delta$ is not an ub.

Thm: $\mathbb{R}$ is an ordered field, extend $\mathbb{Q}$, has lub. property.

FACT: $\mathbb{R}$ is only ordered field with the lub. property.

Consequence: length "$\sqrt{2}$" $= \sup\{1, 1.4, 1.41, 1.414, 1.4147, \cdots\}$
$= 1.4147\cdots$

• More generally, $a^{1/n} \overset{\text{def}}{=} \sup\{r : r^n < a\}$

GREASTEST LOWEST BOUND (glb) or INFIMUM, write $\inf A$

CONSEQS of LUB PROP.

• Archimedean prop of $\mathbb{R}$.
If $x, y \in \mathbb{R}$, $x > 0$, then $\exists$ positive integer $n$ s.t.
$nx > y$.

Equiv: If $x > 0$, $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{n} < x$.

proof: $A = \{nx : n \in \mathbb{N}\}$
(by contradiction) If $A$ were bounded by $y$ (e.g. $nx < y$, $\forall n \in \mathbb{N}$)
So $A$ has an lub, call it $\alpha$.
Then $\alpha - x$ is not u.b. for $A$.
hence $\alpha - x < mx$ for some $m \in \mathbb{N}$
So $\alpha < (m+1)x$ $\rightarrow\!\!\!\!\times$ $\alpha$ is an upper bound for $A$.

• **Thm** : Between $x, y \in \mathbb{R}$, $x < y$.
$\exists\, q \in \mathbb{Q}$ s.t. $x < q < y$.
[ $\mathbb{Q}$ is dense in $\mathbb{R}$ ]

proof : choose $n$ s.t. $\frac{1}{n} < y - x$
consider multiples of $\frac{1}{n}$, these are unbounded.
Choose first multiple s.t. $\frac{m}{n} > x$.
Claim $\frac{m}{n} < y$. If not, then $\frac{m-1}{n} < x$ and $\frac{m}{n} > y$.
But these imply $\frac{1}{n} > y - x$. $\cancel{\times}$

PROPERTIES of SUP:
(a) $r$ is an u.b. for $A$ $\Leftrightarrow$ $\sup A \leq r$.
(b). $\forall\, a \in A$, $a \leq r$ $\Rightarrow$ $\sup A \leq r$.
(c) $\forall\, a \in A$, $a < r$ $\Rightarrow$ $\sup A \leq r$.
(d) $r < \sup A$ $\Rightarrow$ $\exists\, a \in A$ s.t. $r < a \leq \sup A$.
(e) If $A \subset B$, then $\sup A \leq \sup B$
(f) To show $\sup A \leq \sup B$
one strategy : show $\forall\, a \in A$, $\exists\, b \in B$ s.t. $a \leq b$
Thus $\sup A \leq \sup B$.