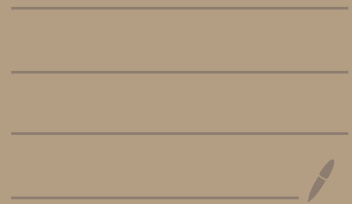


## 04: The Least Upper Bound Property

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• Defined order

arithmetic "+", "x"

• Also,  $\mathbb{R}$  contains  $\mathbb{Q}$  as a subfield.

Associate to  $q \in \mathbb{Q}$ , the cut  $q^* = \{r \in \mathbb{Q} : r < q\}$ .

Check:  $f: \mathbb{Q} \rightarrow \mathbb{R}$  preserve +, x, <

$$q \mapsto q^*$$

Then  $\mathbb{Q}' = \{q^* : q \in \mathbb{Q}\}$  is a subfield of  $\mathbb{R}$ .

Notice: length " $\sqrt{2}$ " sets in  $\mathbb{R}$

$$\text{as } \gamma = \{q : q^2 < 2 \text{ or } q < 0\}$$

check, use def of x, that  $\gamma^2 = 2^*$

•  $\mathbb{R}$  has the least upper bound property!

If  $A$  is collection of cuts with u.b.  $\beta$ .

let  $\gamma = \cup \{d : d \in A\}$ , a subset of  $\mathbb{Q}$ .

check:  $\gamma$  is a cut &  $\gamma = \sup A$

↑

nontriv (bdd).

closed down - no largest member.

$\gamma$  is an ub clearly, since  $\gamma$  contains all  $d \in A$ .

•  $\gamma$  is least upper bound since if  $\delta < \gamma$ ,  $\exists x \in \gamma \setminus \delta$

Then  $x \in$  some  $d \in A$ , not in  $\delta$ , so  $\delta$  is not an ub.

Thm:  $\mathbb{R}$  is an ordered field, extend  $\mathbb{Q}$ , has lub. property.

FACT:  $\mathbb{R}$  is only ordered field with the lub. property.

Consequence: length " $\sqrt{2}$ " =  $\sup \{1, 1.4, 1.41, 1.414, 1.4142, \dots\}$   
= 1.4142...

• More generally,  $a^{1/n} \stackrel{\text{def}}{=} \sup \{r : r^n < a\}$

GREATEST LOWER BOUND (glb) or INFIMUM, write  $\inf A$

CONSEQS of LUB PROP.

• Archimedean prop of  $\mathbb{R}$ .

If  $x, y \in \mathbb{R}$ ,  $x > 0$ , then  $\exists$  positive integer  $n$  s.t.  
 $nx > y$ .

Equip: If  $x > 0$ ,  $\exists n \in \mathbb{N}$  s.t.  $\frac{1}{n} < x$ .

proof:  $A = \{nx : n \in \mathbb{N}\}$

(by contradiction) If  $A$  were bounded by  $y$  (e.g.  $nx < y, \forall n \in \mathbb{N}$ )

So  $A$  has an lub, call it  $d$ .

Then  $d - x$  is not u.b. for  $A$ .

hence  $d - x < mx$  for some  $m \in \mathbb{N}$

So  $d < (m+1)x \rightarrow d$  is an upper bound for  $A$ .

• Thm : Between  $x, y \in \mathbb{R}$ ,  $x < y$ .

$\exists q \in \mathbb{Q}$  s.t.  $x < q < y$ .

[ $\mathbb{Q}$  is dense in  $\mathbb{R}$ ]

proof : choose  $n$  s.t.  $\frac{1}{n} < y - x$

consider multiples of  $\frac{1}{n}$ , these are unbounded.

choose first multiple s.t.  $\frac{m}{n} > x$ .

Claim  $\frac{m}{n} < y$ . If not, then  $\frac{m-1}{n} < x$  and  $\frac{m}{n} > y$ .

But these imply  $\frac{1}{n} > y - x$ . \*

PROPERTIES of SUP:

(a)  $r$  is an u.b. for  $A \iff \sup A \leq r$ .

(b)  $\forall a \in A, a \leq r \implies \sup A \leq r$ .

(c)  $\forall a \in A, a < r \implies \sup A \leq r$ .

(d)  $r < \sup A \implies \exists a \in A$  s.t.  $r < a \leq \sup A$ .

(e) If  $A \subset B$ , then  $\sup A \leq \sup B$

(f) To show  $\sup A \leq \sup B$

one strategy : show  $\forall a \in A, \exists b \in B$  s.t.  $a \leq b$

Thus  $\sup A \leq \sup B$ .