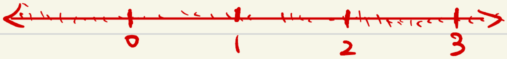
03: Construction of the Reals

CONSTRUCT THE REAL #'s.

DEDEKIND "CUTS" 1872.



x^2 has no solution in $\mathbb{Q}$.

Consider: $A = \{ x \in \mathbb{Q} : x^2 < 2 \}$.

LEAST UPPER BOUND.

Def: Say $E \subset S$ ordered.

If there exists $\beta \in S$ s.t.

for all $x \in E$ we have $x \leq \beta$

then call β an upper bound for E.

say E is bounded above.

(lower bound: replace $\leq$ by $\geq$)

Ex: 2 is an upper bound for A.

$\frac{3}{2}$ is an u.b. for A. (why? If not, $\exists x \in A$, $x > \frac{3}{2}$, then $x^2 > (\frac{3}{2})^2 > 2$ $\times$).

Def: If $\exists \alpha \in S$ s.t.

① α is an upper bound of E.

and ② if $\gamma < \alpha \Rightarrow \gamma$ is not an upper bound of E.

then α is called the least upper bound (lub) of E or supremum of E, write $\alpha = \sup E$

$S = \mathbb{Q}$. Ex: $E = \{ \frac{1}{2}, 1, 2 \}$, $\sup E = 2$

$E = \mathbb{Q}_-$, the neg rational. $\sup E = 0$.

$E = \mathbb{Q}$, $\sup E$ does not exist. (It's unbounded above). $\sup E = +\infty$.

$E = A$, (before), $\sup A$ does not exist

We'll CONSTRUCT $\mathbb{R}$ AND PROVE

Thm : $\mathbb{R}$ is an ordered field, with lub property and $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

A set S has the lub property (satisfies the completeness axiom). if every non-empty subset of S that has an upper bound, also, has a lub in S.

Dedekind : A cut α is a subset of $\mathbb{Q}$ s.t.

① $\alpha \neq \phi, \mathbb{Q}$ [non-trivial]

② If $p \in \alpha$, $q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$ [closed downward]

③ If $p \in \alpha$, then $p < r$ for some $r \in \alpha$. [no largest number].

Ex : A (before) is not a cut.

$\alpha = \mathbb{Q}_-$ is a cut.

$\beta = \{r \in \mathbb{Q} : r \leq 2\}$ is not a cut.

Let $\mathbb{R} \overset{def}{=} \{\alpha : \alpha \text{ is a cut}\}$ some set, show it has structure.

• Def order : Say $\alpha < \beta$ if and only if $\alpha \subsetneq \beta$

- Addition : $\alpha + \beta := \{r+s : r \in \alpha, s \in \beta\}$

check it is cut :

. non-trivial (check).

. closed down : if $p \in \alpha+\beta$, say $q < p = r+s$ is $q \in \alpha+\beta$?

note $q-s < r$, so $q-s \in \alpha$. Then $q = (q-s)+s$ as desired.

Show axioms A1 – A5

add identity $0^* = \mathbb{Q}_-$ Check $\alpha + 0^* = \alpha$

add inverse for α, $\beta = \{ p : \exists r > 0 \text{ s.t. } -p - r \notin \alpha \}$

show $\alpha + \beta = 0^*$

Multiplication

be careful of neg

Def : if $\alpha, \beta \in \mathbb{R}_+$ $\leftarrow$ $(\alpha\beta > 0^*)$

$\alpha\beta := \{ p : p < rs \text{ for some } r \in \alpha, s \in \beta \}$

Let $1^* = \{ q < 1 : q \in \mathbb{Q} \}$