02: Properties of Q (Rational Numbers)

$\frac{1}{2} = \{(1,2), (2,4), (3,6), \dots\}$.

PROPERTIES of $\mathbb{Q}$: ARITHMETIC, ORDER

• Addition? Want: ration that does not depend on representation chosen!

well-defined.

How about $\frac{a}{b} + \frac{c}{d} = \frac{0}{1}$: "well defined but boring".

• Good def'n: $\boxed{\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}}$.

• Multiplication: $\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}}$.

• In what sense does $\mathbb{Q}$ extend $\mathbb{Z}$? Check that $\{\frac{n}{1} : n \in \mathbb{Z}\}$ "belongs in $\mathbb{Z}$"

$\mathbb{Z}$ has an order. Does $\mathbb{Q}$?

• **Def'n** An order on set S is a relation $<$ satisfies ① (trichotomy) If $x, y \in S$.

exactly one at these is true:

$x<y$, $x=y$, $y<x$;

and ② (transitivity): If $x, y, z \in S$,

$x<y$, $y<z \Rightarrow x<z$.

Ex: in $\mathbb{Z}$, say $m<n$ if $n-m$ is positive.

i.e. in the set $\{1, 2, 3, 4, \dots\}$.

Ex: in $\mathbb{Z} \times \mathbb{Z}$, say $(a,b) < (c,d)$

if $a<c$ or $a=c$ and $b<d$ (dictionary order).

Ex: in $\mathbb{Q}$, say $\frac{m}{n}$ is positive if $mn > 0$.

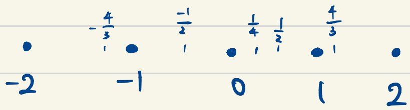
[check: well-defined.]

Then say $\frac{m}{n} < \frac{m'}{n'}$ if $\frac{m'}{n'} - \frac{m}{n}$ is positive.

Write "$y < x$" for $x < y$.

Write "$x \leq y$" for "$x < y$" or "$x = y$".

New picture of $\mathbb{Q}$



$\mathbb{Q}$: Good enough to solve : $5x = 3$, $\Rightarrow x = \frac{3}{5}$

not good enough to solve $x^2 = 2$.

Thm: $x^2 = 2$ has no solution in $\mathbb{Q}$.

proof: (by contradiction).

Assume $x^2 = 2$ has a solution in $\mathbb{Q}$, i.e., say $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$,

and assume [p, q are "in lowest terms", i.e. have no common factors]

So $\left(\frac{p}{q}\right)^2 = 2$, hence $p^2 = 2q^2$

Then p^2 is even (divisible by 2)

Then p is even (because if p is odd, p^2 cannot be even).

So $p = 2m$ for some $m \in \mathbb{Z}$, $p^2 = 4m^2$ and $4m^2 = 2q^2$.

Then $2m^2 = q^2$.

Then q^2 is even, hence q is even.

This contradicts p, q are in lowest terms.

So, $x^2 = 2$ must have no sol'n in $\mathbb{Q}$. #.

$\mathbb{Q}$ is a field.

in $\mathbb{Q}$, 0 element is $\frac{0}{1}$.

1 element is $\frac{1}{1}$.

$\mathbb{Z}$ is not a field. since $\mathbb{Z}$ has no inverse multiplier.

order is presented by field ope.

① $y < z \Rightarrow x+y < x+z$,

② $y < z$, $x > 0 \Rightarrow xy < xz$.

Next time constructing real numbers. $\mathbb{R}$.

They're extend rationals $\mathbb{Q}$.

fill in 'holes' in numbers line.