01: Constructing the Rational Numbers

SETS of RELATIONS

- A set is collection of objects.

  e.g. $S = \{ 1, ☺, ▱, \{1, ☺\} \}$.

  or $S = \{ x : P(x) \text{ is true} \}$.

  $\uparrow$ such that

  $\uparrow$ P is some statement about x.
  eg. "x is less than 2".

Short hand : $x \in S$ means x is in S

$x \notin S$ means x is not in S.

ϕ is the empty set.

$A \subset B$ means "A is a subset of B." which means "if $x \in A$ then $x \in B$".

(or $x \in A \Rightarrow x \in B$)

If $A \subset B$ and $B \not\subset A$, then A is a proper subset of B.

If $A \subset B$ and $B \subset A$, then write $A = B$, else $A \neq B$.

More sets :

union $A \cup B = \{ x : x \in A \ \underline{\text{OR}} \ x \in B \}$.

intersection $A \cap B = \{ x : x \in A \ \underline{\text{AND}} \ x \in B \}$.

complement $A^c = \{ x : x \notin A \}$.

minus $A \setminus B = \{ x : x \in A \text{ and } x \in B \}$.

product $A \times B = \{ (a, b) : a \in A \text{ and } b \in B \}$.

- A (binary) relation R is a subset of $A \times B$.

  If $(a, b) \in R$ write aRb.

Ex: A "is an ancestor of" is a relation on $P \times P$ ← people.

L "likes" $P \times P$

S "is a sibling of" $P \times P$

$<$ "less than" $\mathbb{Z} \times \mathbb{Z}$

- An "equivalent relation" on set S is

  a relation on $S \times S$ s.t. ← "such that"

  - reflexive ① aRa
  - symmetry ② $aRb \Rightarrow bRa$
  - transitive ③ aRb and $bRc \Rightarrow aRc$

  often write $\sim$, $\approx$, $\cong$, etc.

- Aside: A function F from A to B

  is a relation s.t.

  if aFb and aFb' then $b = b'$.

  Write $F(a) = b$

Construction of $\mathbb{Q}$, the rational numbers.

Assume $\mathbb{Z}$, the integers, their multiply, order.

- What is $\mathbb{Q}$? Perhaps, it's the set $\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \}$.

  But we don't know enough what it means.

- Motivation : one over three

  two over six

Write : $(1,3) \sim (2,6)$ is equivalent related pair.

Idea : these belong to some equi class, we'll call "$\frac{1}{3}$".

Let $\mathbb{Q}$ = set of all such equi classes of such pairs $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$.

- we want this pair to extend $\mathbb{Z}$, so that "$\frac{n}{1} \in \mathbb{Q}$ corresponds to $n \in \mathbb{Z}$".

- See that $\mathbb{Q} = \{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \}$.

  where $\frac{m}{n}$ is an equivalent class of (m,n).

  with relation $(p,q) \sim (m,n)$ if $pn = qm$ and $q, n \neq 0$.

- Check $\sim$ is equiv rel'n :

  ① check $(p,q) \sim (p,q)$ ...

  ② check $(p,q) \sim (m,n) \Rightarrow (m,n) \sim (p,q)$

  ③ check $(p,q) \sim (m,n)$ and $(m,n) \sim (a,b)$ than $(p,q) \sim (a,b)$

  [try this : use cancellation law in $\mathbb{Z}$.

  if $ab = ac$ and $a \neq 0$, then $b = c$.]