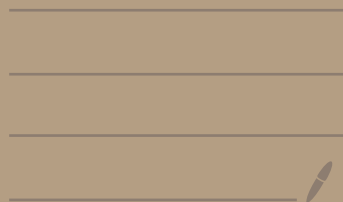


01: Constructing the Rational Numbers



SETS of RELATIONS

- A set is collection of objects.

e.g. $S = \{1, \odot, \square, \{1, \odot\}\}$

or $S = \{x : P(x) \text{ is true}\}$

↑ such that

↑ P is some statement about x.
eg. "x is less than 2".

short hand : $x \in S$ means x is in S

$x \notin S$ means x is not in S.

ϕ is the empty set.

$A \subset B$ means "A is a subset of B" which means "if $x \in A$ then $x \in B$ ".
(or $x \in A \Rightarrow x \in B$)

If $A \subset B$ and $B \not\subset A$, then A is a proper subset of B.

If $A \subset B$ and $B \subset A$, then write $A = B$, else $A \neq B$.

More sets :

union $A \cup B = \{x : x \in A \text{ OR } x \in B\}$

intersection $A \cap B = \{x : x \in A \text{ AND } x \in B\}$

complement $A^c = \{x : x \notin A\}$

minus $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$

product $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

• A (binary) relation R is a subset of $A \times B$.

If $(a, b) \in R$ write aRb .

Ex: A "is an ancestor of" is a relation on $P \times P$
 L "likes" $P \times P$
 S "is a sibling of" $P \times P$
 $<$ "less than" $\mathbb{Z} \times \mathbb{Z}$

↓ people.

• An "equivalent relation" on set S is

a relation on $S \times S$ s.t. \leftarrow "such that"

- reflexive ① aRa

- symmetry ② $aRb \Rightarrow bRa$

- transitive ③ aRb and $bRc \Rightarrow aRc$

often write $\sim, \approx, \cong, \text{etc.}$

• Aside: A function F from A to B

is a relation s.t.

if aFb and aFb' then $b=b'$.

Write $F(a)=b$

Construction of \mathbb{Q} , the rational numbers.

Assume \mathbb{Z} , the integers, their multiply, order.

• What is \mathbb{Q} ? Perhaps, it's the set $\left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$.
↑
But we don't know enough what it means.

• Motivation : \vdash one over three
 \vdash two over six

Write: $(1, 3) \sim (2, 6)$ is equivalent related pair.

Idea: these belong to same equi class, we'll call " $\frac{1}{3}$ ".

Let $\mathbb{Q} =$ set of all such equi classes of such pairs $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$.

• we want this pair to extend \mathbb{Z} , so that " $\frac{n}{1} \in \mathbb{Q}$ corresponds to $n \in \mathbb{Z}$ ".

• See that $\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$.

where $\frac{m}{n}$ is an equivalent class of (m, n) .

with relation $(p, q) \sim (m, n)$ if $pn = qm$ and $q, n \neq 0$.

• Check \sim is equiv rel'n :

① check $(p, q) \sim (p, q)$...

② check $(p, q) \sim (m, n) \Rightarrow (m, n) \sim (p, q)$

③ check $(p, q) \sim (m, n)$ and $(m, n) \sim (a, b)$ then $(p, q) \sim (a, b)$

[try this: use cancellation law in \mathbb{Z} .

if $ab = ac$ and $a \neq 0$, then $b = c$.]