Lecture 25: Taylor's Theorem, Sequence of Functions

TAYLOR'S THEOREM.

Suppose we know $f(a)$. want approximate $f(b)$.

MVT says: $f(b) = f(a) + \underbrace{f'(c)(b-a)}_{\text{"error" not precisely know.}}$ for some $c \in (a,b)$.

- This supposts:

$$f(b) = f(a) + f'(a)(b-a) + \text{error} \leftarrow \text{In fact } \frac{f''(c)}{2}(b-a)^2. \quad (\text{some } c \in (a,b).)$$

More generally, if $P_{n-1}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}$

poly. deg $n-1$.

<u>Taylor's Thm</u>: If $f^{(n-1)}$ contin. on $[a,b]$

$f^{(n)}$ exists on (a,b)

then $P_{n-1}(x)$ approximates $f(x)$

and $f(x) = P_{n-1}(x) + \frac{f^{(n)}(c)}{n!}(x-a)^n$, $c \in (a,b)$.

- When $n=1$, it's MVT
- $P_n(x)$ is $\underbrace{\text{"best" poly approx}}_{\text{same value}}$ of order n of a.

  same value $\left\{ \begin{matrix} f, f', f'', \cdots, f^{(n)} \\ p, p', p'', \cdots, p^{(n)} \end{matrix} \right\}$ at a.

<u>Proof</u>: Clearly, for some number M, $f(b) = P_{n-1}(a) - M(b-a)^n$ —— $(*)$.

Let $g(x) = f(x) - P_{n-1}(x) - M(x-a)^n$.

$g^{(n)}(x) = f^{(n)}(x) - M \cdot n!$

so enough to show $g^{(n)}(c) = 0$ for some $c \in (a,b)$.

Check $g(a) = 0$ (since $f(a) = P_{n-1}(a)$).

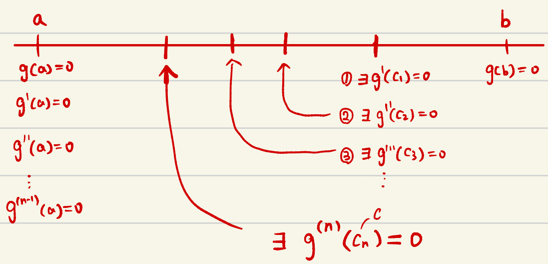
$g'(a) = 0$ $\quad f'(a) = P'_{n-1}(a)$

$g''(a) = 0$ $\quad f''(a) = P''_{n-1}(a)$

$\vdots$

$g^{(n)}(a) = 0$

Also, $g(b) = 0$ by $(*)$.

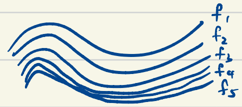


SEQUENCES OF FUNCTIONS.

Q: What does it means for seq of func. to converges?

$f_1(x),\ f_2(x),\ f_3(x), \dots$

A: Pointwise convergence:



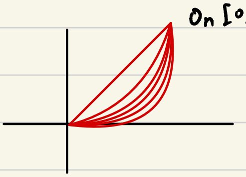
Fix x, does $\{f_n(x)\}$ converges?

If so, pt wise limit $f(x) = \lim_{n\to\infty} f_n(x)$.

Ex ①

$f_n(x) = \frac{x}{n} \xrightarrow[\text{ptw}]{} f(x) = 0$.

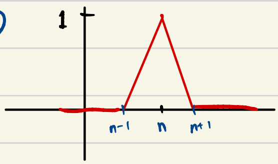
Ex ②



On $[0,1]$

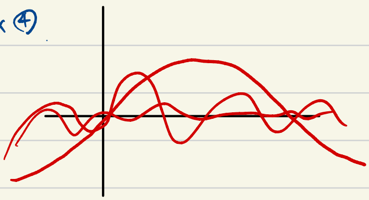
$f(x) = x^n \xrightarrow[\text{pt wise}]{} f(x) = \begin{cases} 1 & \text{at } x = 1 \\ 0 & \text{otherwise} \end{cases}$

Ex ③



$f_n(x) \xrightarrow[P.]{} f(x) = 0$.

Ex ④.



$f_n(x) = \frac{1}{n}\sin(n^2 x) \longrightarrow f(x) = 0$.

Q: What properties preserve by limit pt wise?

Continuity? No ②.

derivatives? No ④.

integral? No ③.

this is usual convergence in matric space $\mathcal{C}_b(E)$. the contin, bounded. func on E $d(f,g) = \|f-g\|$.

Need stronger notion, let $\|f\| = \sup_{x\in E} \|f\|$

Def: (Uniformly convergence) Say $f_n \xrightarrow{u} f$ (f_n converges uniformly to f") on E

if $\forall \varepsilon > 0$, $\exists N$

some N works for all $x \in E$ s.t. $n \geq N \Rightarrow \|f_n - f\| < \varepsilon$. ~ (can draw ε-ribbon about the limit f all f_n is eventually stays in ribbon.)

• Fact $\mathcal{C}_b(E)$ is complete.

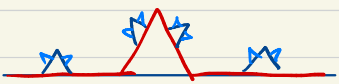
so we have Cauchy Critirion.

Thm: $f_n \to f$ on E $\iff$ $\forall \varepsilon > 0$, $\exists N$ s.t.

$\forall n, m > N$, s.t. $\forall x \in E$

$|f_n(x) - f_m(x)| < \varepsilon$.

Ex: $f_n : [0,1] \to \mathbb{R}^2$.



like so, see it's Cauchy.

so it converges.

what's its limit?

(It's contin.) by

Thm: If $f_n \xrightarrow{u} F$, f_n contin. then f contin.

proof: idea:

$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$

Fix x.

$\forall \varepsilon > 0$, ① choose f_n so $\|f_n(x) - f\| < \varepsilon/3$

② Then f_n conti, $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon/3$

So $\forall \varepsilon > 0$, we find $\delta > 0$ s.t. $|f(x) - f(y)| < |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$

$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$

$= \varepsilon$.

Appl. Thm: $\exists f : [0,1] \to [0,1]^2$ box, that is space-filling