The 2x2 Exchange Economy

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2019/6/13

(Calculus 4, Applications)

Road Map for 2x2 Exchange Economy

- Pareto Efficiency Allocation (PEA)
  - Cannot make one better off without hurting others
- Walrasian (Price-taking) Equilibrium (WE)
  - When Supply Meets Demand
  - Focus on Exchange Economy First
- 1st Welfare Theorem:
  - Any WE is PEA (Adam Smith Theorem)
- 2nd Welfare Theorem:
  - Any PEA can be supported as a WE with transfers

2x2 Exchange Economy

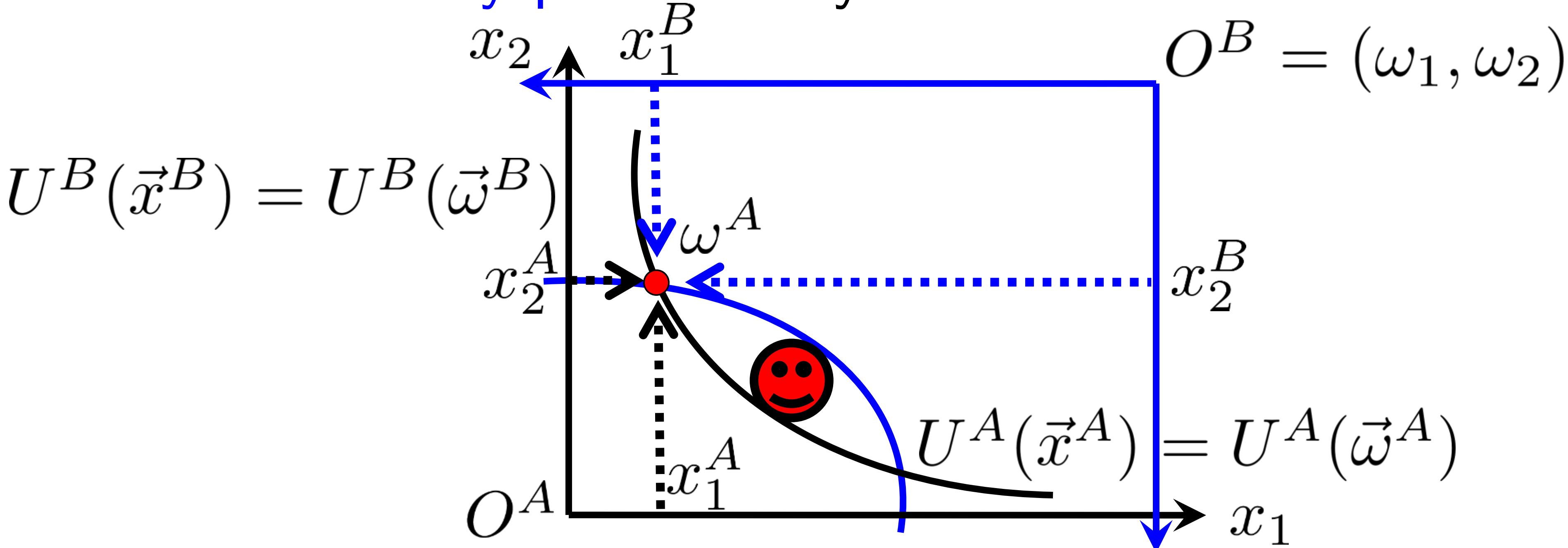
- 2 Commodities: Good 1 and 2
- 2 Consumers: Alex and Bev - $h = A, B$
  - Endowment: $\vec{\omega}^h = (\omega_1^h, \omega_2^h)$, $\omega_i = \omega_i^A + \omega_i^B$
  - Consumption Set: $\vec{x}^h = (x_1^h, x_2^h) \in \mathbb{R}_+^2$
  - Strictly Monotonic Utility:

$$U^h(\vec{x}^h) = U^h(x_1^h, x_2^h), \quad \frac{\partial U^h}{\partial x_i^h}(\vec{x}^h) > 0$$

- Edgeworth Box
  - These consumers could be representative agents, or literally TWO people (bargaining)

Pareto Efficiency

- A feasible allocation is Pareto efficient if
- there is no other feasible allocation that is
- strictly preferred by at least one consumer
- and is weakly preferred by all consumers.

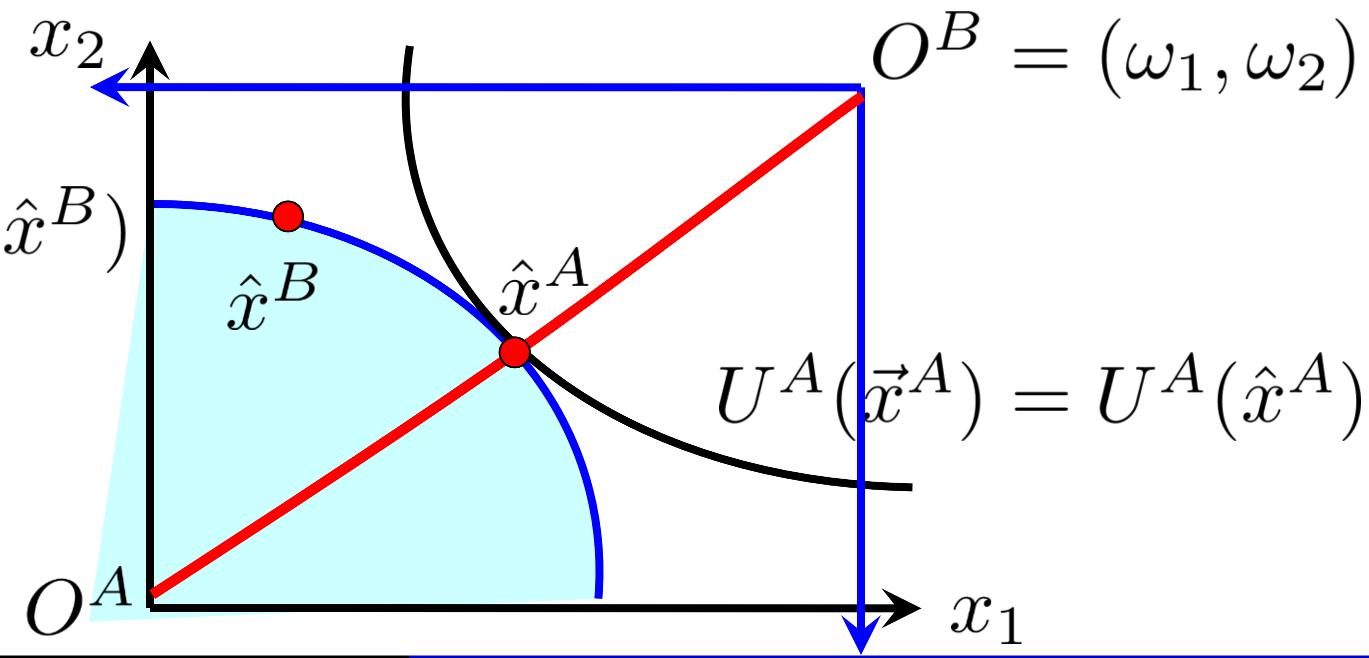


Pareto Efficient Allocations

For $\vec{\omega} = (\omega_1, \omega_2)$, consider

$$\max_{\vec{x}^A, \vec{x}^B} \{U^A(\vec{x}^A) | U^B(\vec{x}^B) \geq U^B(\hat{x}^B), \vec{x}^A + \vec{x}^B \leq \vec{\omega}\}$$

Need $MRS^A(\hat{x}^A) = MRS^B(\hat{x}^A)$ (interior solution)



PEA with Cobb-Douglas Utility

$$\max_{x,y} U^A(x,y) = x^\alpha y^{1-\alpha}$$

$$\text{s.t. } U^B = (\overline{x}-x)^\beta (\overline{y}-y)^{1-\beta} \geq \underline{U}^B$$

$$\mathcal{L} = x^\alpha y^{1-\alpha} - \lambda \cdot \left[\underline{U}^B - (\overline{x}-x)^\beta (\overline{y}-y)^{1-\beta}\right]$$

FOC: (for interior solutions)

$$\frac{\partial \mathcal{L}}{\partial x} = \alpha \cdot \frac{y^{1-\alpha}}{x^{1-\alpha}} - \beta\lambda \cdot \frac{(\overline{y}-y)^{1-\beta}}{(\overline{x}-x)^{1-\beta}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = (1-\alpha) \cdot \frac{x^\alpha}{y^\alpha} - (1-\beta)\lambda \cdot \frac{(\overline{x}-x)^\beta}{(\overline{y}-y)^\beta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \underline{U}^B - (\overline{x}-x)^\beta (\overline{y}-y)^{1-\beta} = 0$$

PEA with Cobb-Douglas Utility

Meaning of FOC: $MRS^A = MRS^B$

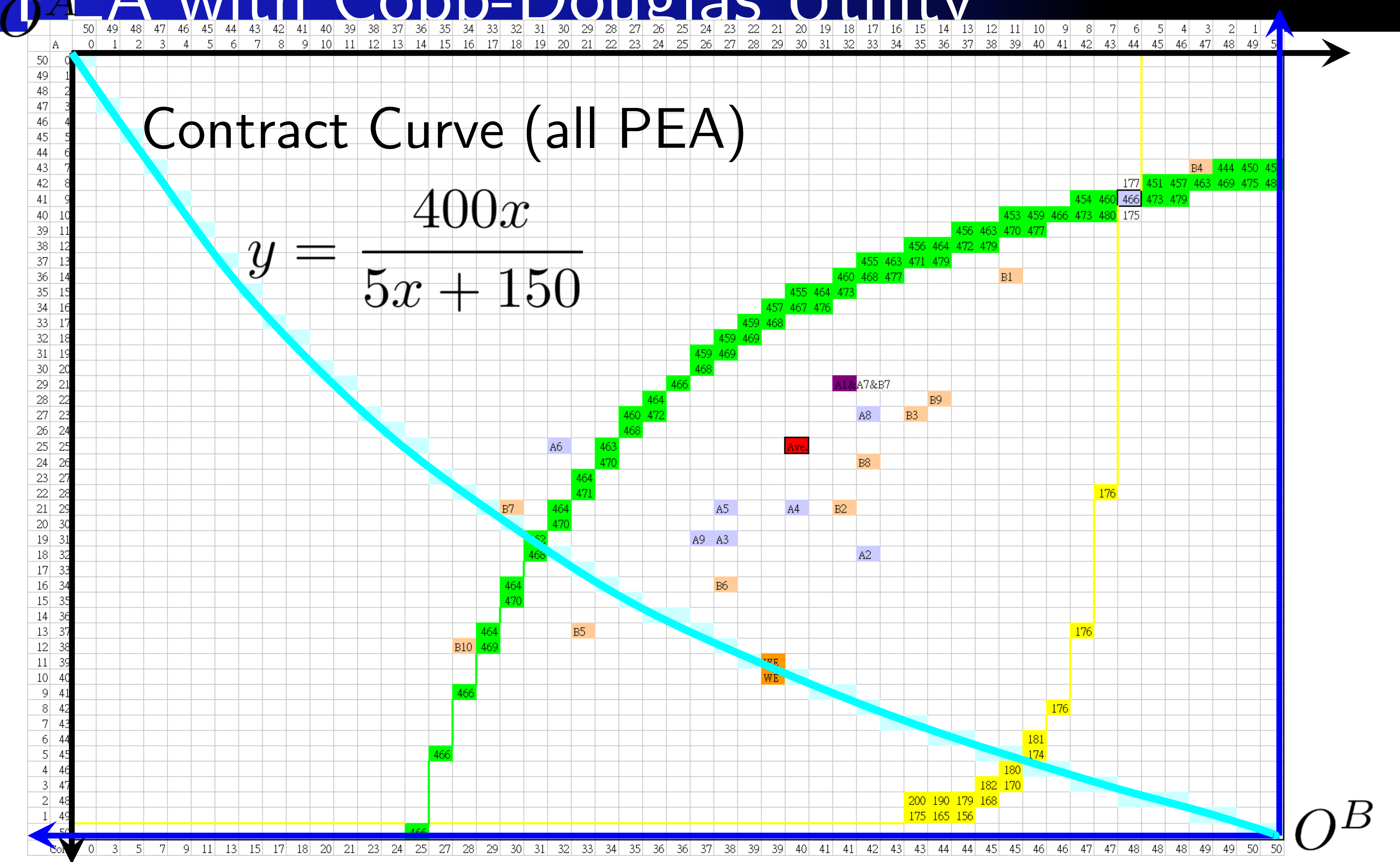
$$\lambda = \frac{\alpha \cdot \dfrac{y^{1-\alpha}}{x^{1-\alpha}}}{\beta \cdot \dfrac{(\overline{y}-y)^{1-\beta}}{(\overline{x}-x)^{1-\beta}}} = \frac{(1-\alpha) \cdot \dfrac{x^{\alpha}}{y^{\alpha}}}{(1-\beta) \cdot \dfrac{(\overline{x}-x)^{\beta}}{(\overline{y}-y)^{\beta}}}$$

$$\Rightarrow \alpha \cdot y \cdot (1-\beta) \cdot (\overline{x}-x) = \beta \cdot (\overline{y}-y) \cdot (1-\alpha) \cdot x$$

$$y = \frac{\beta(1-\alpha) \cdot \overline{y} \cdot x}{\alpha(1-\beta)(\overline{x}-x) + \beta(1-\alpha) \cdot x} = \frac{\gamma \overline{y} \cdot x}{(\gamma - 1)x + \overline{x}}$$

$$= \frac{\frac{8}{3} \cdot 50 \cdot x}{\frac{5}{3}x + 50} = \frac{400x}{5x + 150} \qquad \alpha = 0.6, \quad \beta = 0.8 \qquad \gamma = \frac{\beta(1-\alpha)}{\alpha(1-\beta)}$$

PEA with Cobb-Douglas Utility



Walrasian Equilibrium - 2x2 Exchange Economy

- All Price-takers: Price vector $\vec{p} \geq 0$
- 2 Consumers: Alex and Bev - $h \in \mathcal{H} = \{A, B\}$
  - Endowment: $\vec{\omega}^h = (\omega_1^h, \omega_2^h)$, $\omega_i = \omega_i^A + \omega_i^B$
  - Consumption Set: $\vec{x}^h = (x_1^h, x_2^h) \in \mathbb{R}_+^2$
  - Wealth: $W^h = \vec{p} \cdot \vec{\omega}^h$
- Market Demand: $\vec{x}(\vec{p}) = \sum_h \vec{x}^h(\vec{p}, \vec{p} \cdot \vec{\omega}^h)$

  (Solution to consumer problem)
- Vector of Excess Demand: $\vec{z}(\vec{p}) = \vec{x}(\vec{p}) - \vec{\omega}$
  - Vector of total Endowment: $\vec{\omega} = \sum_h \vec{\omega}^h$

Definition: Market Clearing Prices

- Let Excess Demand for Commodity j be $z_j(\vec{p})$
- The Market for Commodity j Clears if
  - Excess Demand = 0 or Price = 0 (and ED < 0)
    - Excess demand = shortage; negative ED means surplus

$$z_j(\vec{p}) \leq 0 \text{ and } p_j \cdot z_j(\vec{p}) = 0$$

- Why is this important?

1. Walras Law
   - The last market clears if all other markets clear
2. Market clearing defines Walrasian Equilibrium

Local Non-Satiation Axiom (LNS)

For any consumption bundle $\vec{x} \in C \subset \mathbb{R}^n$
and any δ-neighborhood $N(\vec{x}, \delta)$ of $\vec{x}$,
there is some bundle $\vec{y} \in N(\vec{x}, \delta)$ s.t. $\vec{y} \succ_h \vec{x}$

- LNS implies consumer must spend all income
- If not, we have $\vec{p} \cdot \vec{x}^h < \vec{p} \cdot \vec{\omega}^h$ for optimal $\vec{x}^h$
- But then there exist δ-neighborhood $N(\vec{x}^h, \delta)$
- In the budget set for sufficiently small $\delta > 0$
- LNS $\Rightarrow \vec{y} \in N(\vec{x}^h, \delta), \vec{y} \succ_h \vec{x}^h$, $\vec{x}^h$ is not optimal!

Walras Law

- For any price vector $\vec{p}$, the market value of excess demands must be zero, because:

$$\vec{p} \cdot \vec{z}(\vec{p}) = \vec{p} \cdot (\vec{x} - \vec{\omega}) = \vec{p} \cdot \left(\sum_h (\vec{x}^h - \vec{\omega}^h) \right)$$

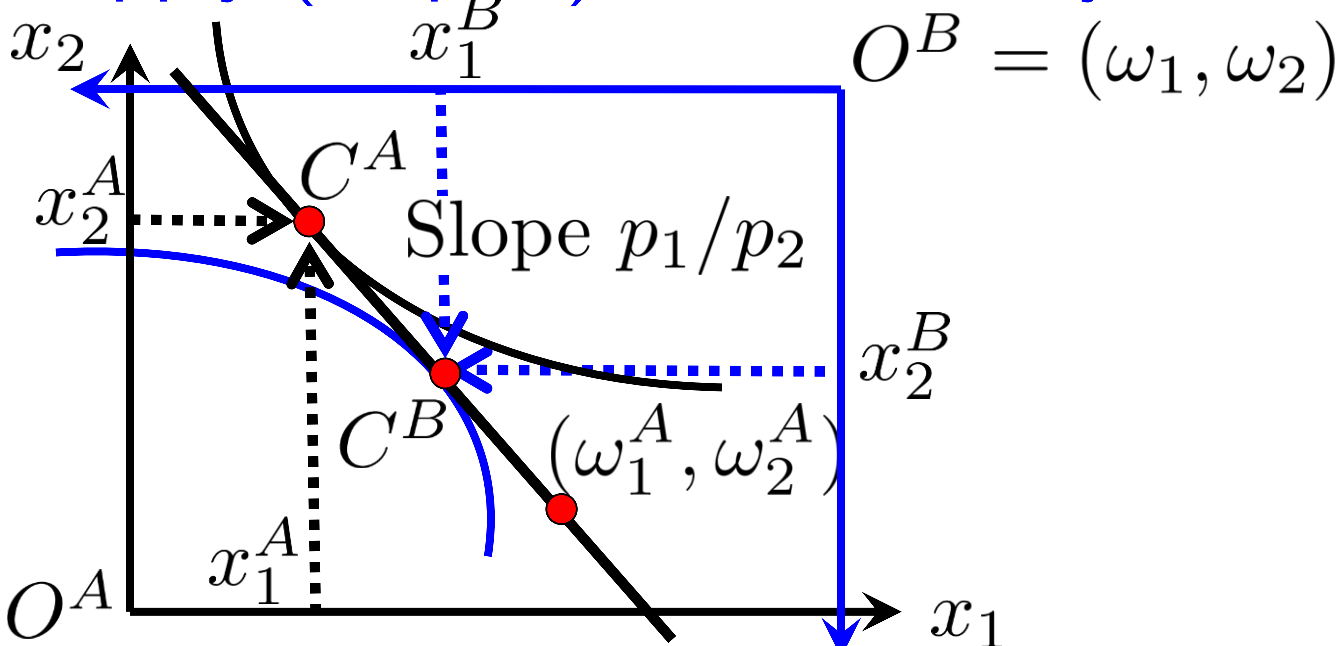
$$= \sum_h (\vec{p} \cdot \vec{x}^h - \vec{p} \cdot \vec{\omega}^h) = 0 \text{ by LNS}$$

$$= p_1 z_1(\vec{p}) + p_2 z_2(\vec{p}) = 0$$

- If one market clears, so must the other.

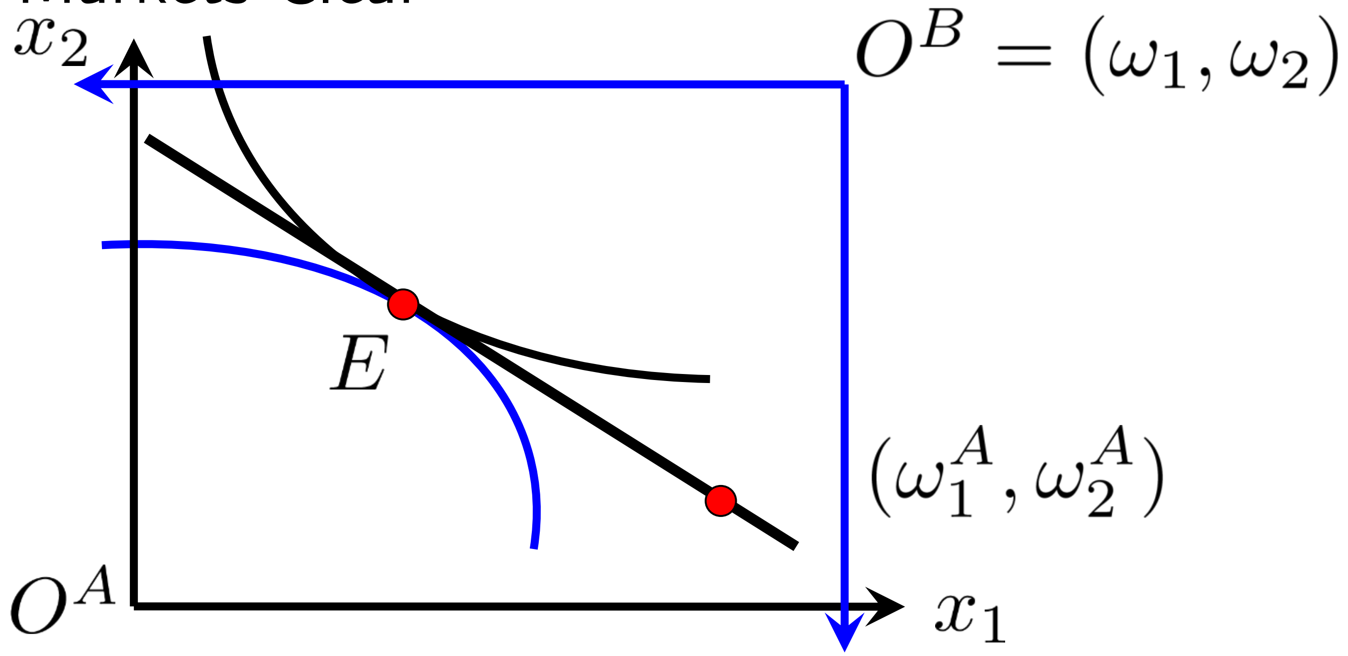
Definition: Walrasian Equilibrium

- The price vector $\vec{p} \geq \vec{0}$ is a Walrasian Equilibrium price vector if all markets clear.
  - WE = price vector!!!
- EX: Excess supply (surplus) of commodity 1...



Definition: Walrasian Equilibrium

- Lower price for commodity 1 if excess supply
  - Until Markets Clear



- Cannot raise Alex's utility without hurting Bev
  - Hence, we have FWT...

First Welfare Theorem: WE → PEA

- If preferences satisfy LNS, then a Walrasian Equilibrium allocation (in an exchange economy) is Pareto efficient.
- Sketch of Proof:

1. Any weakly (strictly) preferred bundle must cost at least as much (strictly more) as WE
2. Markets clear
   → Pareto preferred allocation not feasible

Walrasian Equilibrium: Consumer A Problem

$$\max_{x,y} U^A(x,y) = x^\alpha y^{1-\alpha}$$

$$\text{s.t. } P_x \cdot x + P_y \cdot y \le I^A = P_x \cdot \omega_x^A + P_y \cdot \omega_y^A$$

$$\mathcal{L} = x^\alpha y^{1-\alpha} - \lambda \cdot \left[P_x \cdot x + P_y \cdot y - I^A\right]$$

FOC: (for interior solutions)

$$\frac{\partial \mathcal{L}}{\partial x} = \alpha \cdot \frac{y^{1-\alpha}}{x^{1-\alpha}} - \lambda \cdot P_x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = (1-\alpha) \cdot \frac{x^\alpha}{y^\alpha} - \lambda \cdot P_y = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = P_x \cdot x + P_y \cdot y - I^A = 0$$

Walrasian Equil.: Consumer Optimal Choice

Meaning of FOC: $MRS^A = \frac{P_x}{P_y}$

$$\frac{P_x}{P_y} = \frac{\alpha}{1-\alpha} \cdot \frac{y}{x} \quad \Rightarrow x = \frac{\alpha}{1-\alpha} \cdot \frac{P_y}{P_x} \cdot y$$

$$\Rightarrow I^A = P_x \cdot x + P_y \cdot y = \frac{P_y}{1-\alpha} \cdot y$$

$$\Rightarrow y_A^* = (1-\alpha) \cdot \frac{I^A}{P_y}, \quad x_A^* = \alpha \cdot \frac{I^A}{P_x}$$

Similarly, $y_B^* = (1-\beta) \cdot \frac{I^B}{P_y}$, $x_B^* = \beta \cdot \frac{I^B}{P_x}$

The Walrasian Equilibrium: Markets Clear

$$x_A^* = \alpha \cdot \frac{P_x\omega_x^A + P_y\omega_y^A}{P_x} = \alpha\omega_x^A + \alpha \cdot \frac{P_y}{P_x} \cdot \omega_y^A$$

$$x_B^* = \beta \cdot \frac{P_x\omega_x^B + P_y\omega_y^B}{P_x} = \beta\omega_x^B + \beta \cdot \frac{P_y}{P_x} \cdot \omega_y^B$$

Markets Clear: $x_A^* + x_B^* = \omega_x^A + \omega_x^B$

$$\Rightarrow \left(\alpha \cdot \omega_y^A + \beta \cdot \omega_y^B\right) \cdot \frac{P_y}{P_x} = (1-\alpha)\omega_x^A + (1-\beta)\omega_x^B$$

$$\frac{P_y}{P_x} = \frac{(1-\alpha)\omega_x^A + (1-\beta)\omega_x^B}{\alpha \cdot \omega_y^A + \beta \cdot \omega_y^B}$$

Walras. Equil. in Edgeworth Box Experiment

$$\alpha = 0.6, \beta = 0.8$$

$$(\omega_x^A, \omega_y^A) = (44, 9), \quad (\omega_x^B, \omega_y^B) = (6, 41),$$

$$\Rightarrow \frac{P_y}{P_x} = \frac{(1-\alpha)\omega_x^A + (1-\beta)\omega_x^B}{\alpha \cdot \omega_y^A + \beta \cdot \omega_y^B}$$

$$= \frac{(0.4)44 + (0.2)6}{0.6 \cdot 9 + 0.8 \cdot 41} = \frac{17.6 + 1.2}{5.4 + 32.8} = \frac{94}{191}$$

$$\Rightarrow \frac{P_x}{P_y} = \frac{191}{94} \approx 2.032$$

Edgeworth Box Experiment

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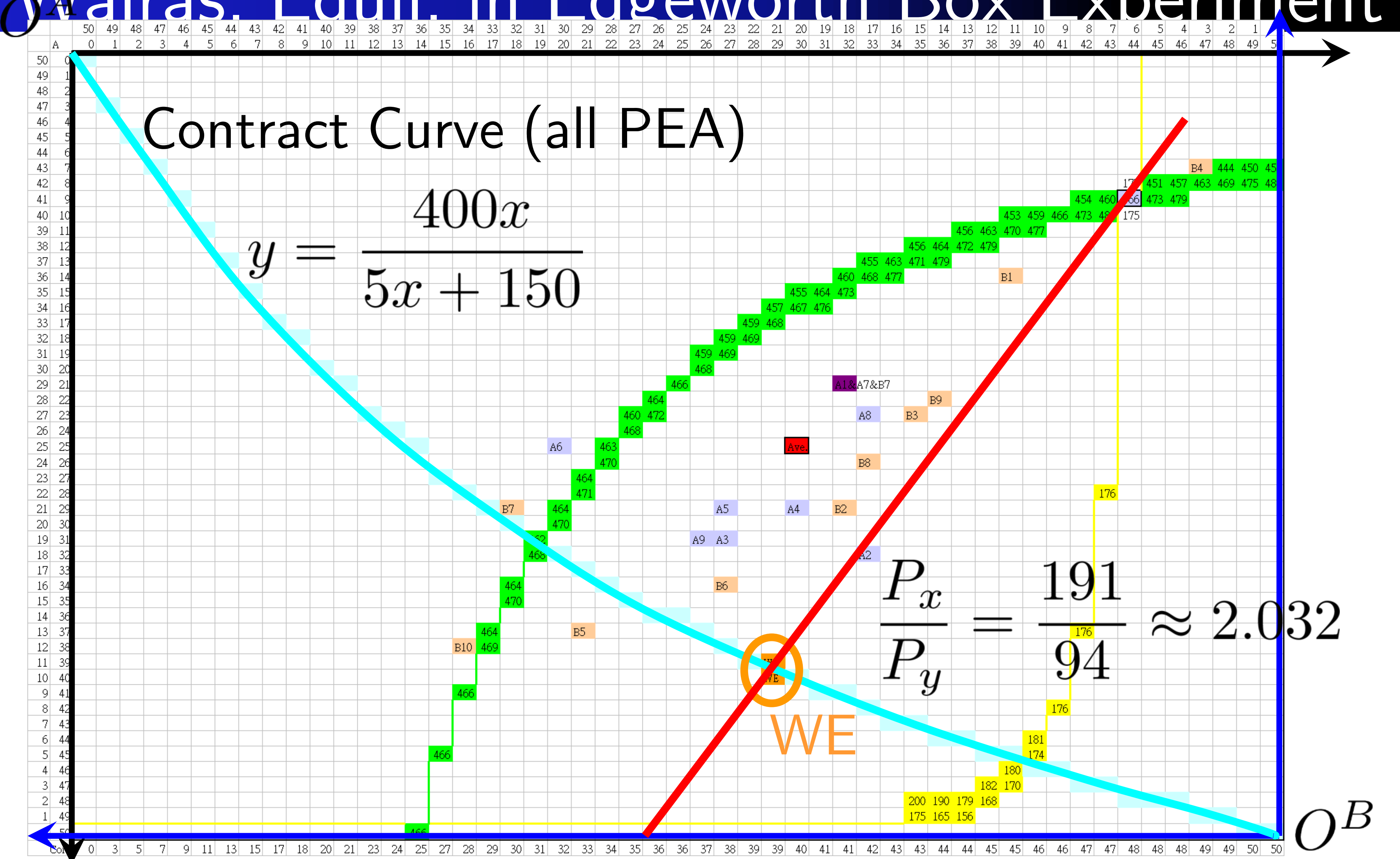
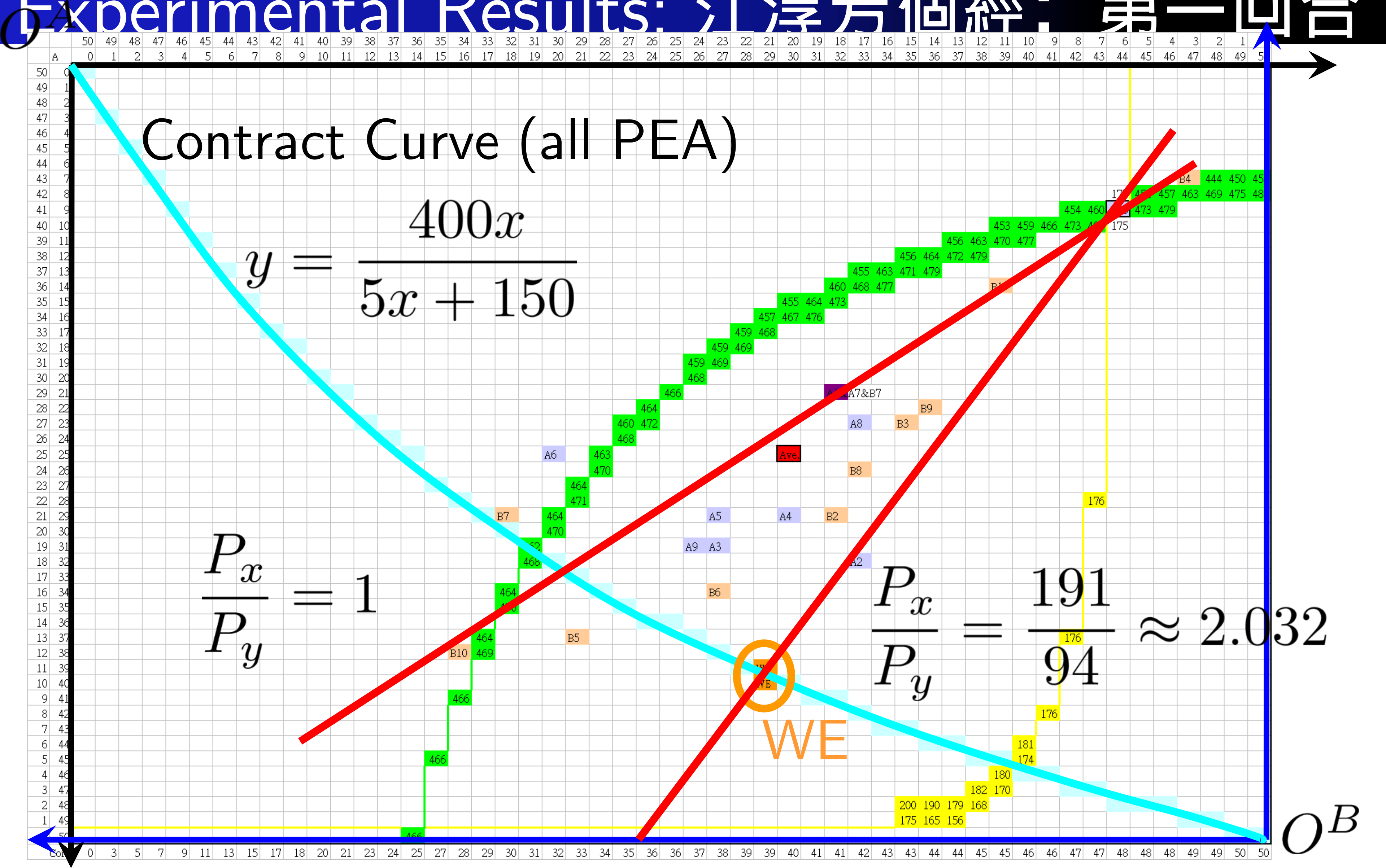
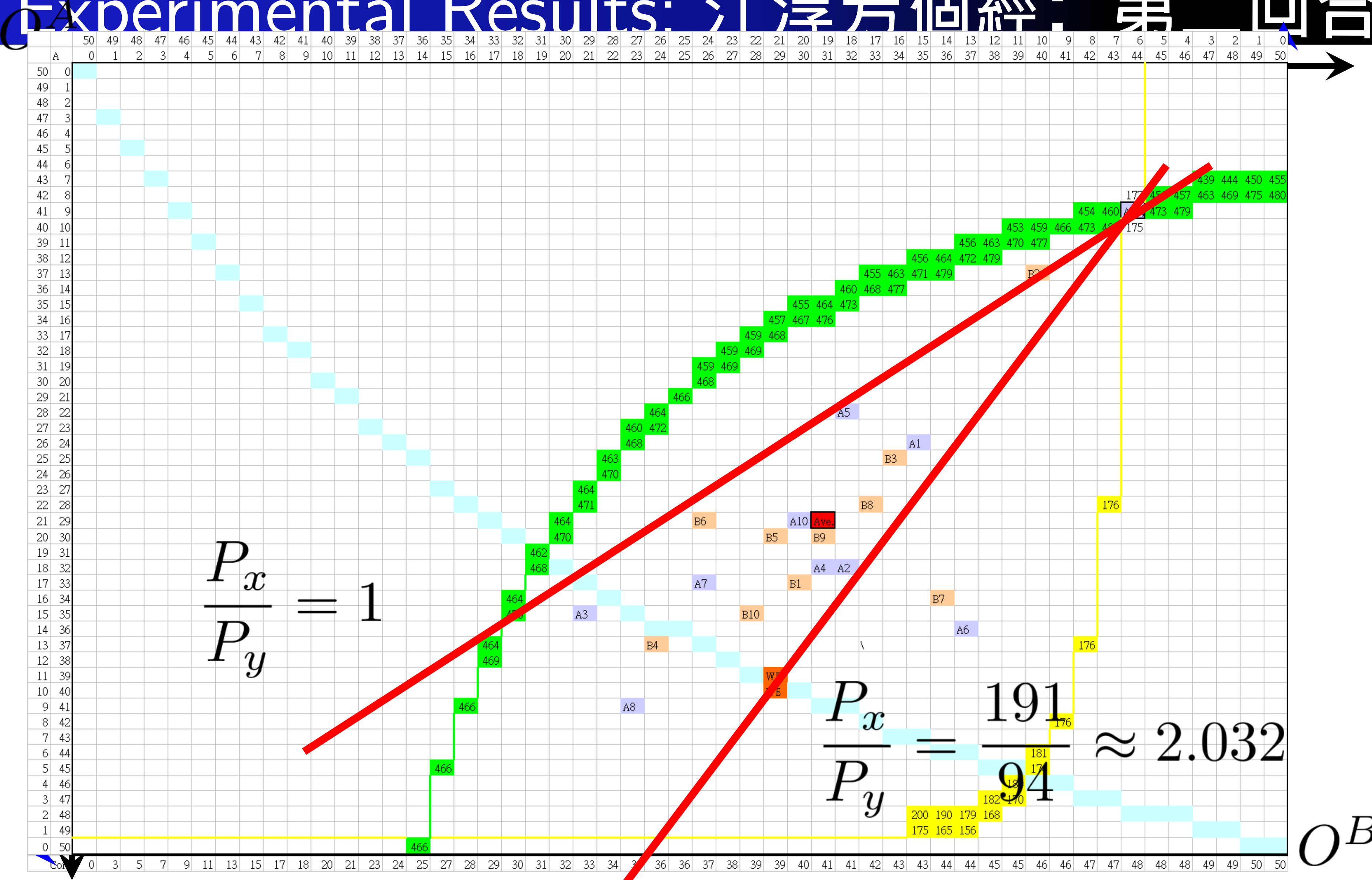
Walras. Equil. in Edgeworth Box Experiment



Experimental Results: 江淳芳個經：第一回合



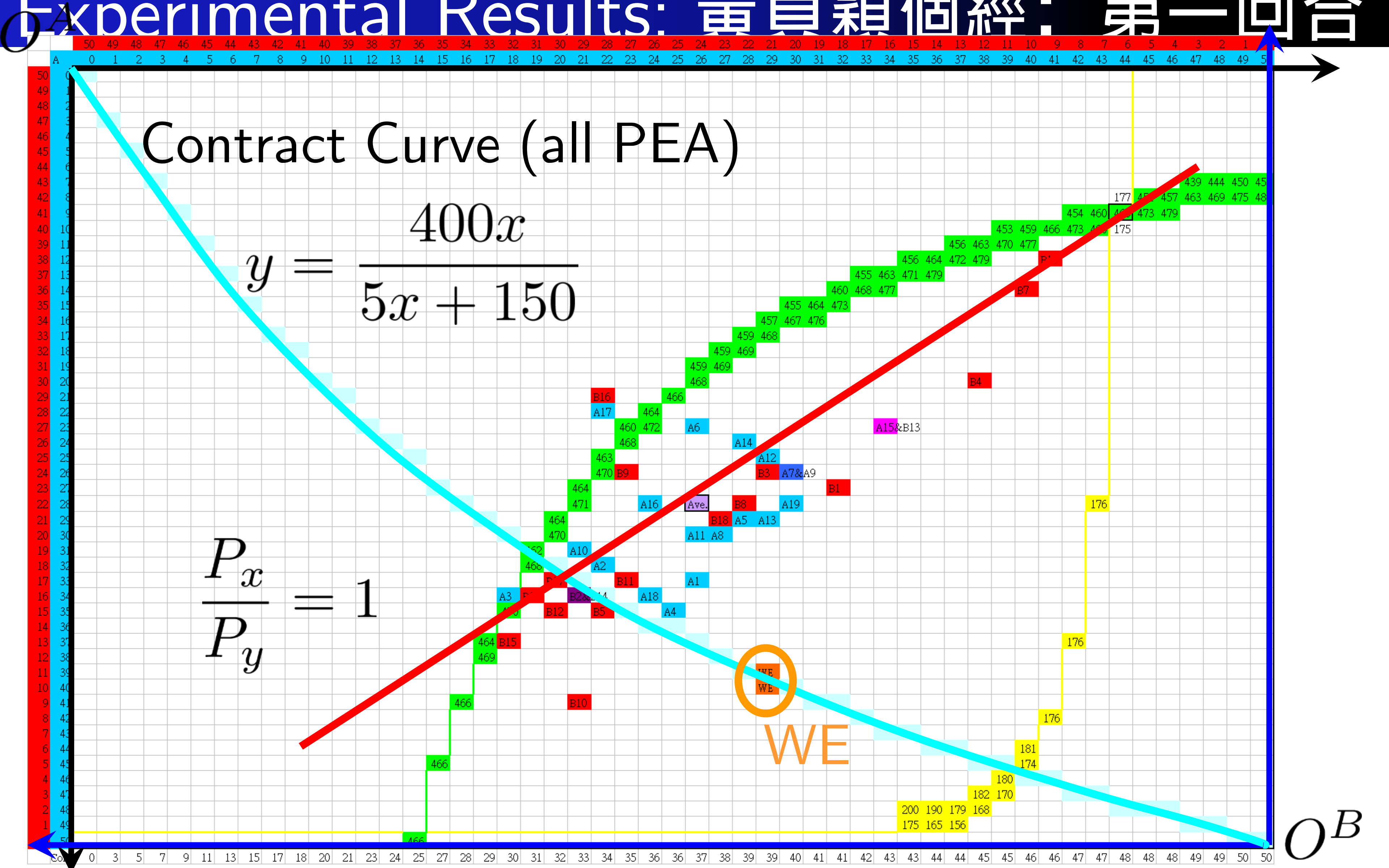
Experimental Results: 汀渟芳個經: 第二回合



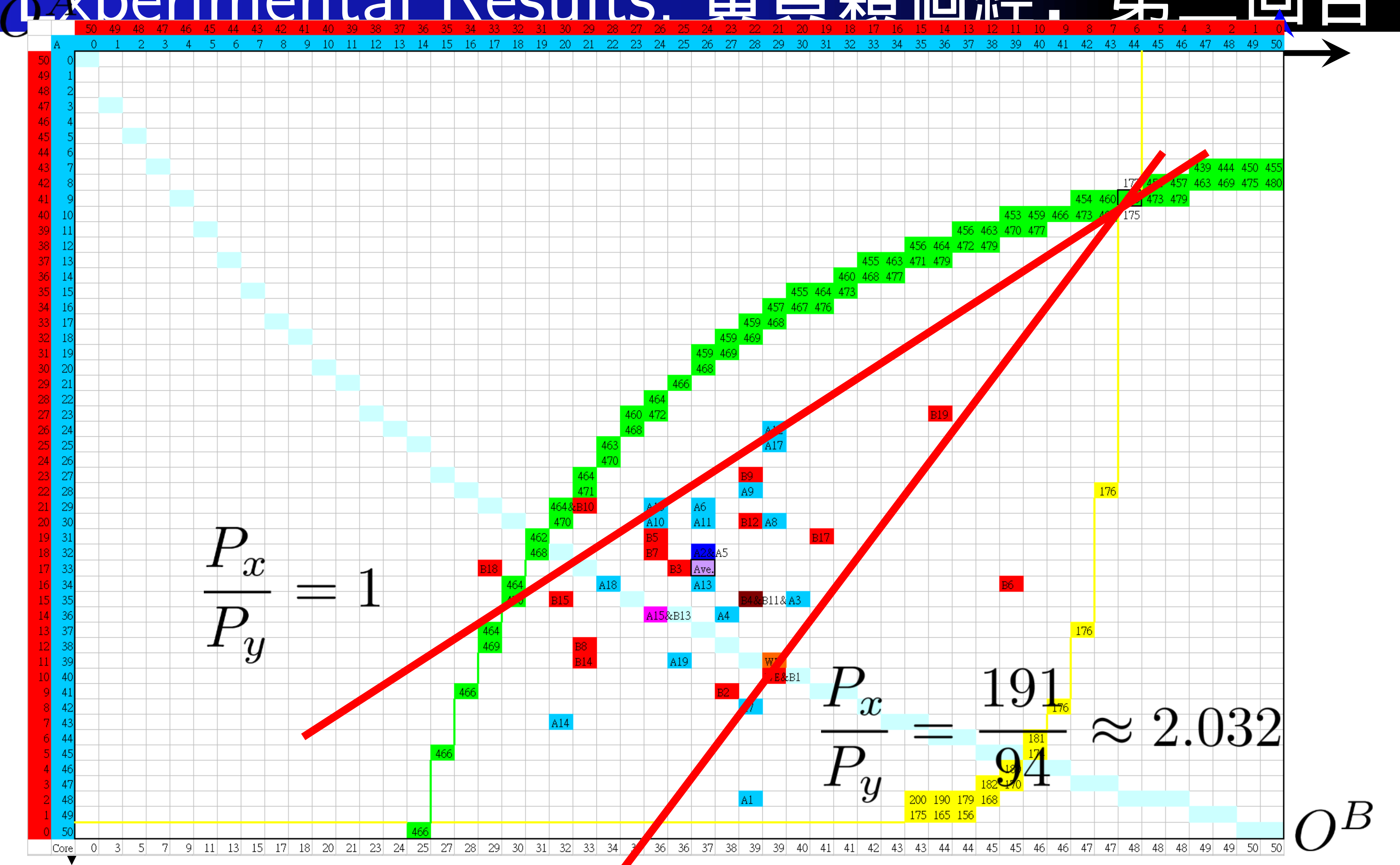
$$\frac{P_x}{P_y} = 1$$

$$\frac{P_x}{P_y} = \frac{191}{94} \approx 2.032$$

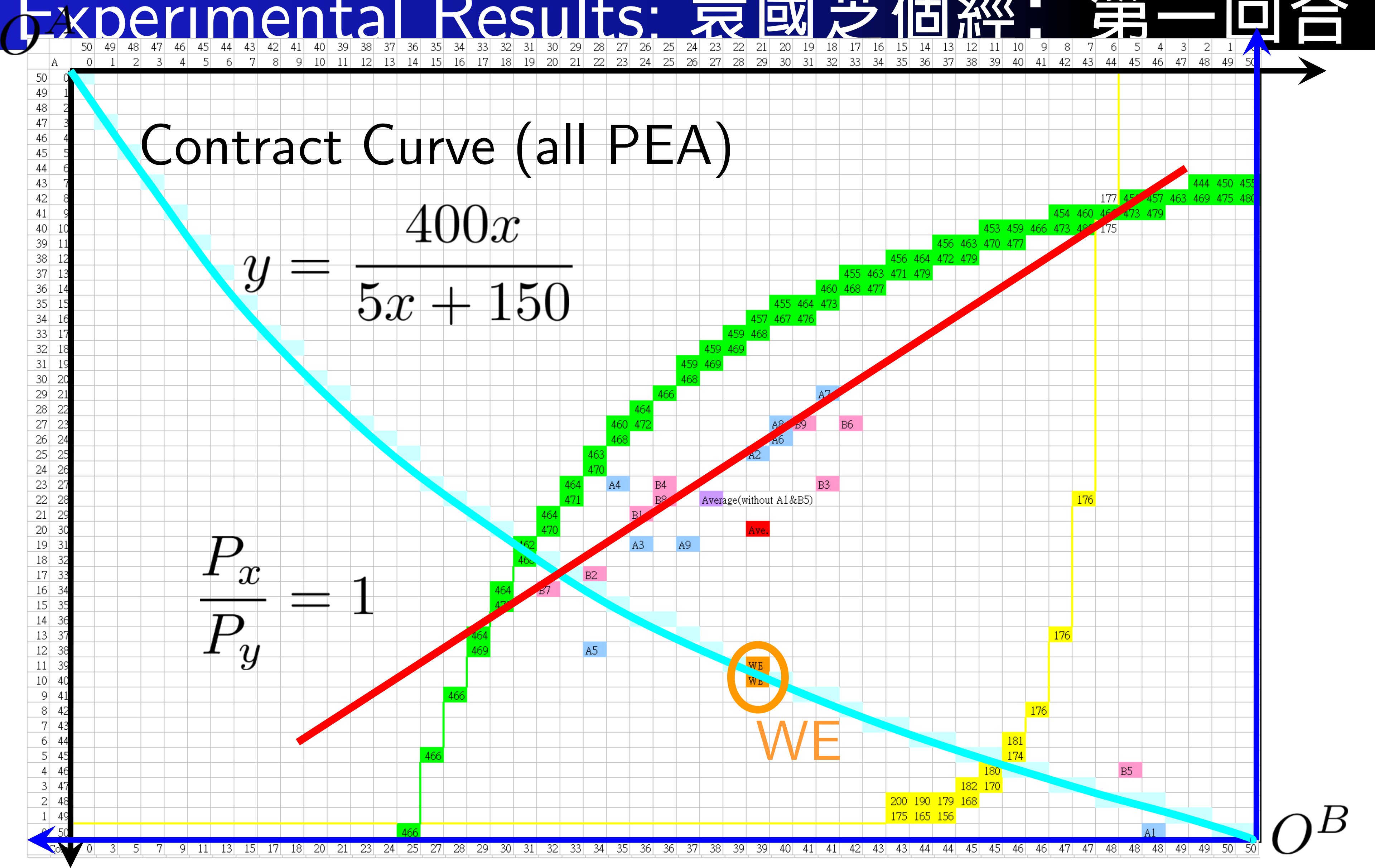
Experimental Results: 黃貞穎個經: 第一回合



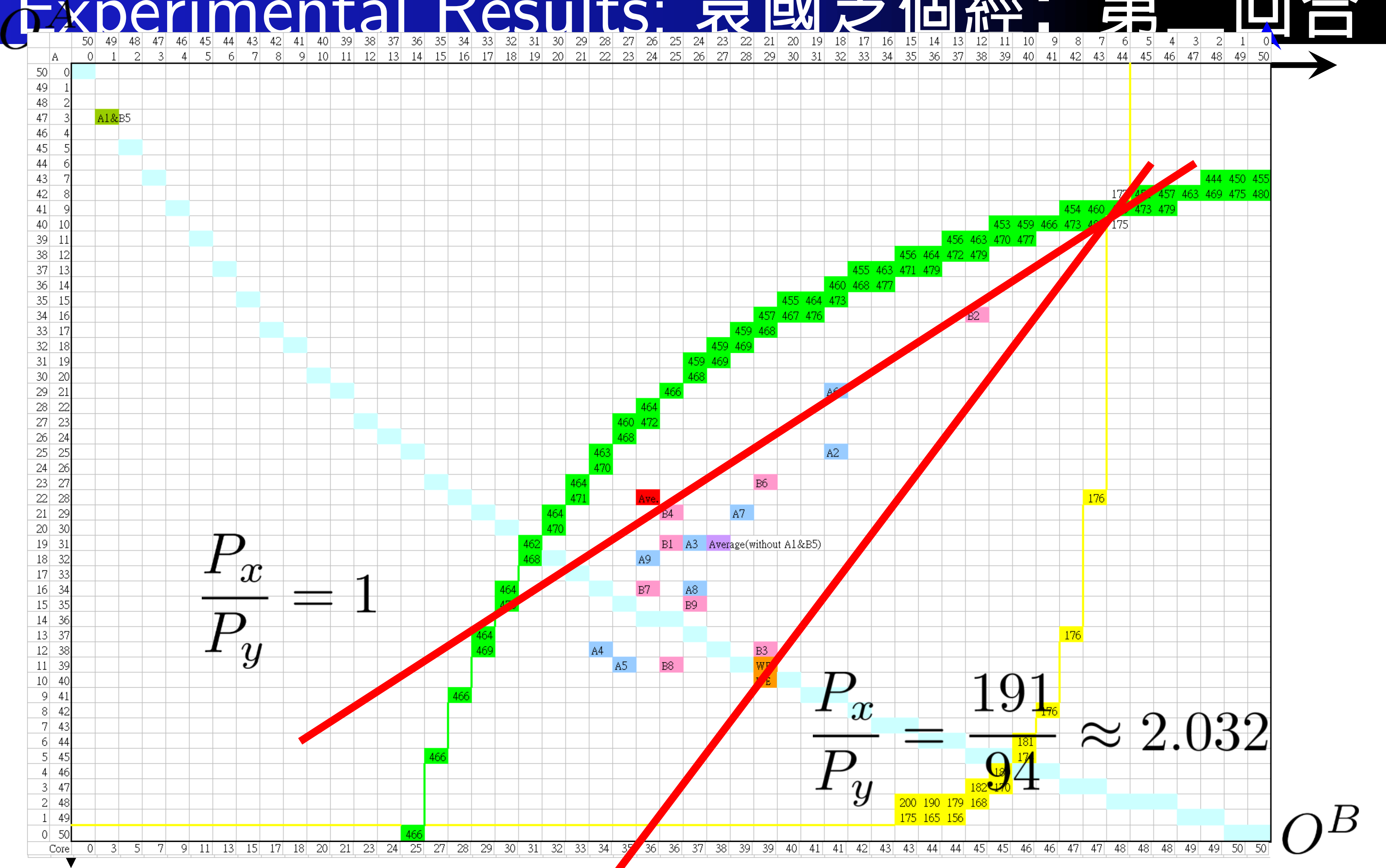
Experimental Results: 黃貞穎個經: 第二回合



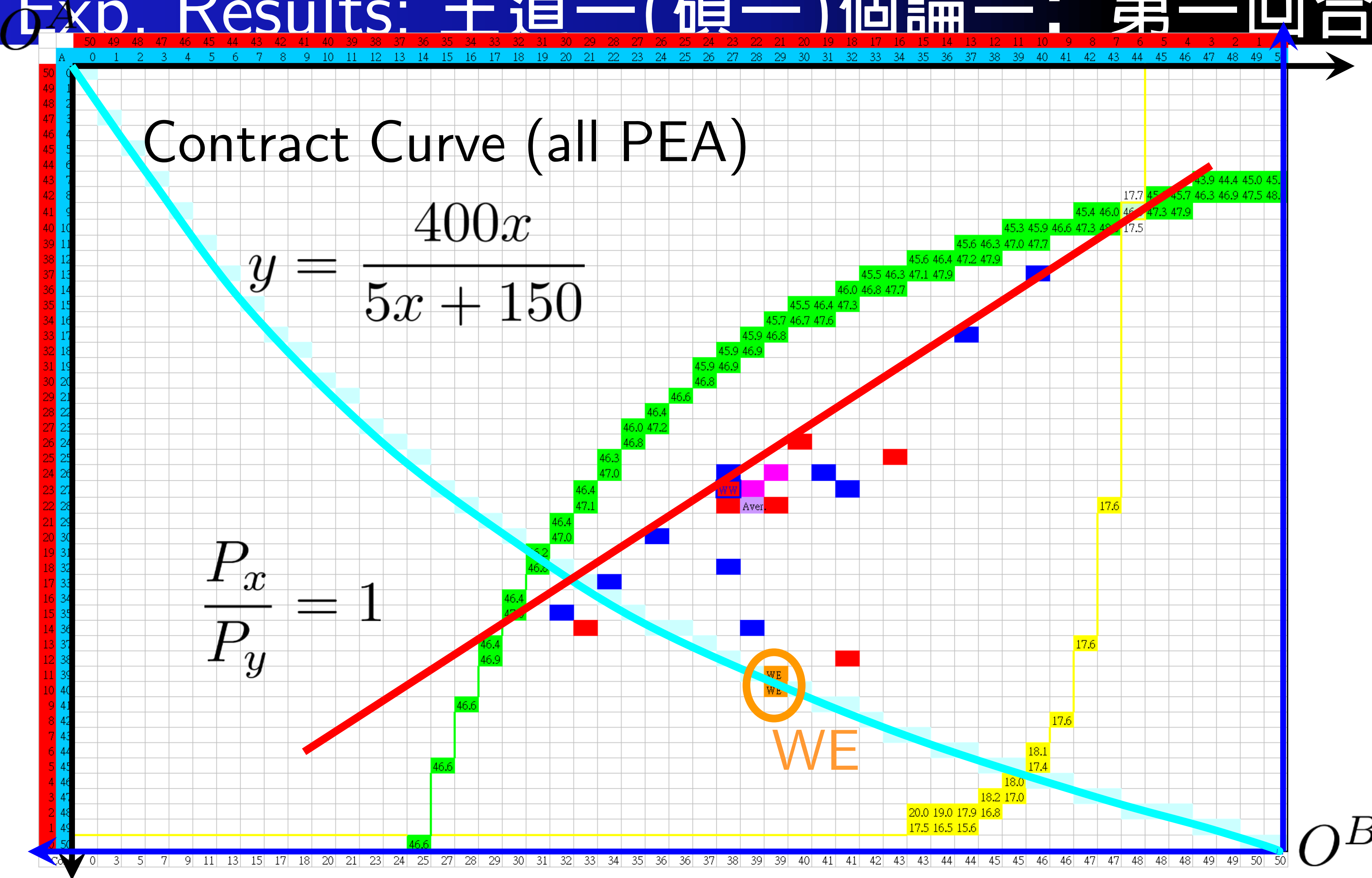
Experimental Results: 袁國芝個經：第一回合



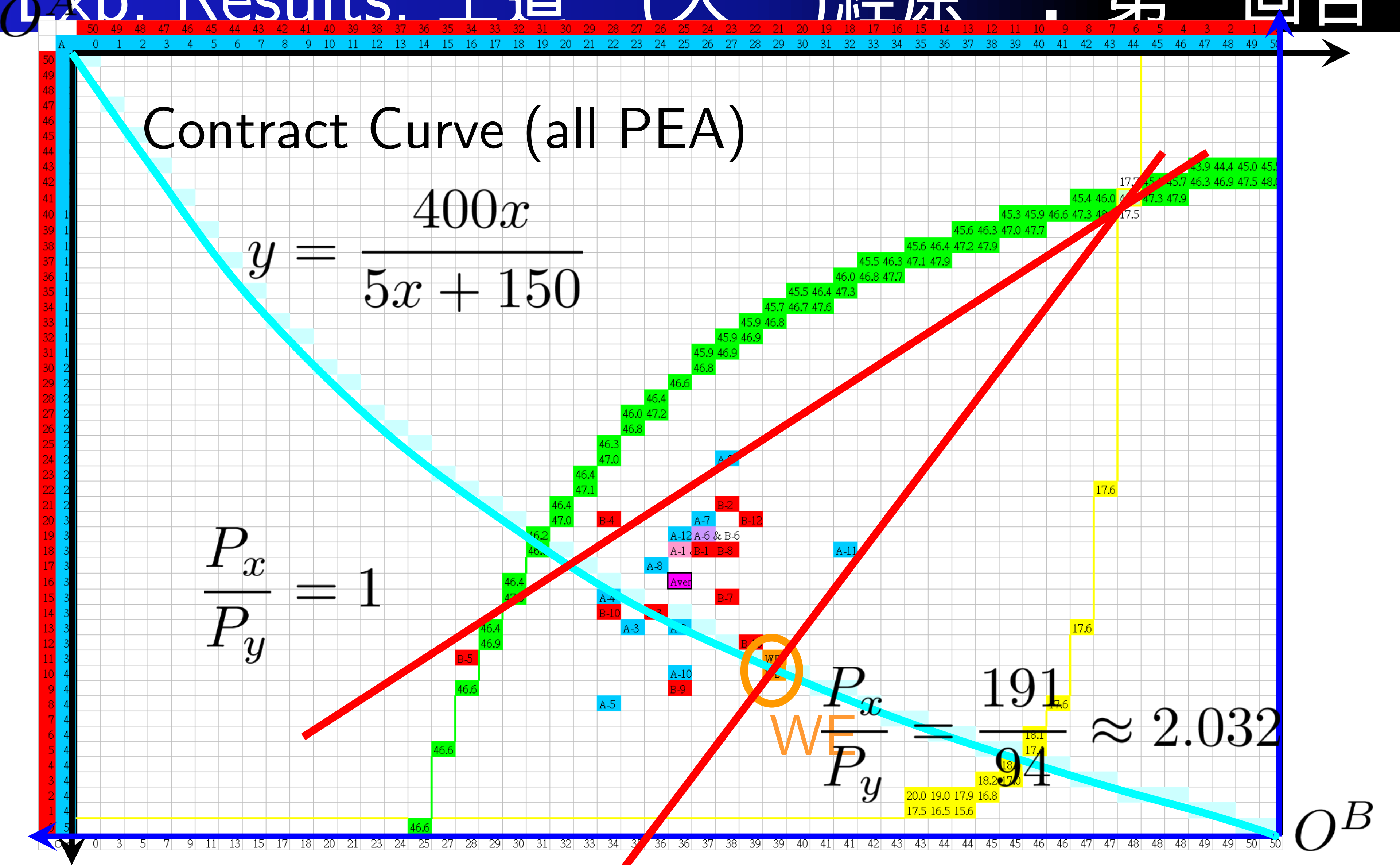
Experimental Results: 袁國芝個經：第二回合



Exp. Results: 王道一(碩一)個論一： 第一回合



Exp. Results: 王道一(大一)經原一：第一回合



What Have We Learned?

- Bilateral trade happens in the Eye
- Prices converge toward WE prices
- Final positions converge toward core and WE
  - Average closer in 2nd round; variance decreases
- Still a lot of noise (but does not effect results)
- Markets work without full information (Hayek)
- What provided the force of competition?
  - Existence of perfect substitute (other A and Bs)
- How can we get further converge?
  - Experience? Larger space? Other trading rules?