Constraint Qualifications

Joseph Tao-yi Wang
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(Calculus 4, 19.5)

What Happens when NDCQ Fails?

- Critical Points where NDCQ Fails are also candidates for the solution!
  - Need to check those points (as well as FOC)
- Can we incorporate this into our Lagrangian as well (just like binding/non-binding cases)?
  - Yes! Add multiplier μ_0 to the objective function!
- Actually, there are other CQ's we can use
  - NDCQ is only one of them...

Theorem 19.10 (CQ: Equality Constraint)

- Suppose f, h are C^1 functions on $\boldsymbol{R}^2$
- $\vec{x}^* = (x_1^*, x_2^*)$ solves

$$\max\left\{f(x_1, x_2) \middle| h(x_1, x_2) = c\right\}$$

- Form the Lagrangian (with multiplier μ_0 for f)

$$\mathcal{L} = \mu_0 f(x_1, x_2) - \mu_1 \left[h(x_1, x_2) - c\right]$$

- There exists $\vec{\mu}^* = (\mu_0^*, \mu_1^*)$ such that
  a) $(\mu_0^*, \mu_1^*) \neq (0, 0)$
  b) $\mu_0^* = 0 \text{ or } 1$
  c) FOC satisfied at $(x_1^*, x_2^*, \mu_0^*, \mu_1^*)$

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- FOC satisfied at $(x_1^*, x_2^*, \mu_0^*, \mu_1^*)$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \mu_0 \frac{\partial f}{\partial x_1}(x_1, x_2) - \mu_1 \frac{\partial h}{\partial x_1}(x_1, x_2) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \mu_0 \frac{\partial f}{\partial x_2}(x_1, x_2) - \mu_1 \frac{\partial h}{\partial x_2}(x_1, x_2) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu_1} = c - h(x_1, x_2) = 0$$

Example 19.9 (CQ: Equality Constraint)

- $\max \left\{ f(x,y) = x \middle| h(x,y) = x^3 + y^2 = 0 \right\}$
- Form Lagrangian $\mathcal{L} = \mu_0 x - \mu_1 [x^3 + y^2 - 0]$
- There exists $(\mu_0^*, \mu_1^*) \neq (0,0)$ s.t. $\mu_0^* = 0 \text{ or } 1$
- FOC: $\dfrac{\partial \mathcal{L}}{\partial x} = \mu_0 - \mu_1 [3x^2] = 0$

$$\frac{\partial \mathcal{L}}{\partial y} = -\mu_1 [2y] = 0$$

(No solution w/o multiplier μ_0)

$$\frac{\partial \mathcal{L}}{\partial \mu_1} = -[x^3 + y^2] = 0$$

Theorem 19.11 (Fritz John Theorem)

- Suppose f, $g_1, \ldots, g_k$ are C^1 functions on $\boldsymbol{R}^n$
- Let $\vec{x}^* = (x_1^*, \cdots, x_n^*)$ be a local maximizer of

$$\max\left\{f(x_1, \cdots, x_n)\middle| g_1(x_1, \cdots, x_n) \le b_1, \cdots, g_k(x_1, \cdots, x_n) \le b_k\right\}$$

- Form the Lagrangian (with multiplier λ_0 for f)

$$\mathcal{L} = \lambda_0 f(x_1, \cdots, x_n) - \lambda_1[g_1(x_1, \cdots, x_n) - b_1] - \cdots - \lambda_k[g_k(x_1, \cdots, x_n) - b_k]$$

- There exists $\vec{\lambda}^* = (\lambda_0^*, \lambda_1^*, \cdots, \lambda_k^*)$ such that

Theorem 19.11 (Fritz John Theorem)

$$\mathcal{L} = \lambda_0 f(x_1, \cdots, x_n) - \lambda_1 [g_1(x_1, \cdots, x_n) - b_1] - \cdots - \lambda_k [g_k(x_1, \cdots, x_n) - b_k]$$

a) $\dfrac{\partial \mathcal{L}}{\partial x_1}(\vec{x}^*, \vec{\lambda}^*) = 0, \cdots, \dfrac{\partial \mathcal{L}}{\partial x_n}(\vec{x}^*, \vec{\lambda}^*) = 0$

b) $\lambda_1^* [g_1(\vec{x}^*) - b_1] = 0, \cdots, \lambda_k^* [g_k(\vec{x}^*) - b_k] = 0$

c) $\lambda_1^* \geq 0, \cdots, \lambda_k^* \geq 0$

d) $g_1(\vec{x}^*) - b_1 \leq 0, \cdots, g_k(\vec{x}^*) - b_k \leq 0$

e) $\lambda_0^* = 0 \text{ or } 1$ (If NDCQ holds, set $\lambda_0^* = 1$!)

f) $(\lambda_0^*, \lambda_1^*, \cdots, \lambda_k^*) \neq (0, 0, \cdots, 0)$

Theorem 19.11 (Fritz John Theorem)

- If NDCQ fails, row vectors linearly dependent:

$$Dg_i = \nabla g_i = \left(\frac{\partial g_i}{\partial x_1}(\vec{x}^*), \cdots, \frac{\partial g_i}{\partial x_n}(\vec{x}^*) \right)$$

- So, for binding constraints, exists $(a_1, \cdots, a_{k_0}) \neq (0, \cdots, 0)$

$$a_1 \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\vec{x}^*) \\ \vdots \\ \frac{\partial g_1}{\partial x_n}(\vec{x}^*) \end{pmatrix} + \cdots + a_{k_0} \begin{pmatrix} \frac{\partial g_{k_0}}{\partial x_1}(\vec{x}^*) \\ \vdots \\ \frac{\partial g_{k_0}}{\partial x_n}(\vec{x}^*) \end{pmatrix} = \vec{0}$$

Set $(\lambda_0^*, \lambda_1^*, \cdots, \lambda_k^*) = (0, a_1, \cdots, a_{k_0}, 0, \cdots, 0)$!

Generalized Example 18.9

$$\max \ f(x,y) = xy$$
$$\text{s.t.} \ (x+y-I)^3 \leq 0$$
$$x \geq 0, y \geq 0$$

- Same as constraint $x + y - I \leq 0$
- But does NDCQ hold at $(x^*, y^*) = \left(\frac{I}{2}, \frac{I}{2}\right)$?
- For $g(x,y) = (x+y-I)^3$

$$\frac{\partial g}{\partial x} = 3(x+y-I)^2$$
$$\frac{\partial g}{\partial y} = 3(x+y-I)^2$$

$$\vec{\nabla} g\left(\frac{I}{2}, \frac{I}{2}\right) = (0,0)$$

So NDCQ fails!

Fritz John Thm on Generalized Example 18.9

- $\tilde{\mathcal{L}} = \lambda_0 xy - \lambda_1 [x + y - I]^3$ has FOC:

$$\frac{\partial \tilde{\mathcal{L}}}{\partial x} = \lambda_0 y - 3\lambda_1 [x + y - I]^2 \leq 0, x \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial x} = 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial y} = \lambda_0 x - 3\lambda_1 [x + y - I]^2 \leq 0, y \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial y} = 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_1} = -[x + y - I]^3 \geq 0, \lambda_1 \cdot \left(-[x + y - I]^3\right) = 0$$

- Hence, $(x^*, y^*) = (0, 0)$ or $\lambda_0^* = 0 (\Rightarrow \lambda_1^* > 0)$

$$\lambda_1^* = 0 (\Rightarrow \lambda_0^* = 1) \text{ or } [x^* + y^* - I]^3 = 0$$

Generalize Example 18.9

- Need to check $(x^*, y^*, \lambda_0^*, \lambda_1^*) = (0, 0, 0, \lambda_1^*)$
  - But this yields zero utility! Not Max!
- Or, $(x^*, y^*, \lambda_0^*, \lambda_1^*) = (x^*, I - x^*, 1, 0)$

  that solves $\max \; f(x, y) = xy = x(I - x)$

$$\text{s.t. } x + y - I = 0$$

$$x \geq 0, y \geq 0$$

- I.e. $(x^*, y^*) = \left(\frac{I}{2}, \frac{I}{2}\right)$

Implicit Function Theorem (Th'm 15.7)

- $F_1, \cdots, F_m : \mathbf{R^{m+n}} \to \mathbf{R^1}$ are C^1 functions
- Let $(\vec{y}^*, \vec{x}^*) = (y_1, \cdots, y_m, x_1, \cdots, x_n)$ solve

$$\begin{cases} F_1(y_1, \cdots, y_m, x_1, \cdots, x_n) = & 0 \\ \vdots & \vdots \\ F_m(y_1, \cdots, y_m, x_1, \cdots, x_n) = & 0 \end{cases}$$

If $\det \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(\vec{y}^*, \vec{x}^*) & \cdots & \frac{\partial F_1}{\partial y_m}(\vec{y}^*, \vec{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1}(\vec{y}^*, \vec{x}^*) & \cdots & \frac{\partial F_m}{\partial y_m}(\vec{y}^*, \vec{x}^*) \end{pmatrix} \neq 0$

Implicit Function Theorem (Th'm 15.7)

Then, exists C^1 functions $y_1 = f_1(x_1, \cdots, x_n)$

$$\vdots$$

$$y_m = f_m(x_1, \cdots, x_n)$$

defined on a ball B around $\vec{x}^*$ such that

$$\begin{cases} F_1(f_1(\vec{x}), \cdots, f_m(\vec{x}), \vec{x}) = 0, & y_1^* = f_1(\vec{x}^*) \\ \vdots & \vdots \\ F_m(f_1(\vec{x}), \cdots, f_m(\vec{x}), \vec{x}) = 0, & y_m^* = f_m(\vec{x}^*) \end{cases}$$

For all $\vec{x} = (x_1, \cdots, x_n) \in B$

Implicit Function Theorem (Th'm 15.7)

- And $\frac{\partial f_k}{\partial x_h}(\vec{y}^*, \vec{x}^*) = \frac{\partial y_k}{\partial x_h}(\vec{y}^*, \vec{x}^*)$ is computed by

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_h} \\ \vdots \\ \frac{\partial y_m}{\partial x_h} \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x_h} \\ \vdots \\ \frac{\partial F_m}{\partial x_h} \end{pmatrix}$$

Theorem 19.12 (Alternative CQ's)

- Suppose f, $g_1, \ldots, g_k$ are C^1 functions on $\boldsymbol{R}^n$
- Let $\vec{x}^* = (x_1^*, \cdots, x_n^*)$ be a local maximizer of

$$\max \left\{ f(x_1, \cdots, x_n) \middle| g_1(x_1, \cdots, x_n) \le b_1, \cdots, g_k(x_1, \cdots, x_n) \le b_k \right\}$$

- Notation: Constraints $g_1, \ldots, g_{k_0}$ binds

$$g_1(x_1^*, \cdots, x_n^*) = b_1, \cdots, g_{k_0}(x_1^*, \cdots, x_n^*) = b_{k_0}$$

- Constraints $g_{k_0+1}, \ldots, g_k$ do not binds

$$g_{k_0+1}(x_1^*, \cdots, x_n^*) < b_{k_0+1}, \cdots, g_k(x_1^*, \cdots, x_n^*) < b_k$$

Theorem 19.12 (Alternative CQ's)

- If constraints $g_1, \ldots, g_{k_0}$ satisfies one of these:

a) NDCQ: Jacobian matrix has maximum rank k_0

$$\begin{pmatrix} \nabla g_1 \\ \vdots \\ \nabla g_{k_0} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\vec{x}^*) & \cdots & \frac{\partial g_1}{\partial x_n}(\vec{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{k_0}}{\partial x_1}(\vec{x}^*) & \cdots & \frac{\partial g_{k_0}}{\partial x_n}(\vec{x}^*) \end{pmatrix}_{k_0 \times n}$$

- Or, row vectors

$$Dg_i = \nabla g_i = \left(\frac{\partial g_i}{\partial x_1}(\vec{x}^*), \cdots, \frac{\partial g_i}{\partial x_n}(\vec{x}^*) \right)$$

are linearly independent

Theorem 19.12 (Alternative CQ's)

b) Karush-Kuhn-Tucker CQ: For $\vec{v} \in \mathbf{R}^n$ such that $Dg_i(\vec{x}^*)(\vec{v}) \leq 0$, Exists $\epsilon > 0$ and $\alpha : [0, \epsilon) \to \mathbf{R}^n$ (C^1 curve)

1. $\alpha(0) = \vec{x}^*$
2. $\alpha'(0) = \vec{v}$
3. $g_i(\alpha(t)) \leq b_i, t \in [0, \epsilon), i = 1, \cdots, k$

c) Slater CQ: Exists a ball U about $\vec{x}^*$ in $\mathbf{R}^n$ such that $g_1, \ldots, g_{k_0}$ are convex functions on U and there exists $\vec{z} \in U$ so that each $g_i(\vec{z}) < b_i$

d) Constraints $g_1, \ldots, g_{k_0}$ are concave functions

e) Constraints $g_1, \ldots, g_{k_0}$ are linear functions

Definition: Concave/Convex Functions

- A set U is a convex set if any line segment between two points in the set is also in the set.

$$\vec{x}, \vec{y} \in U \Rightarrow l(\vec{x}, \vec{y}) = \left\{ t\vec{x} + (1-t)\vec{y} \middle| 0 \le t \le 1 \right\}$$

- f is a concave function on U if $\forall \vec{x}, \vec{y} \in U, t \in [0,1]$

$$f\left(t\vec{x} + (1-t)\vec{y}\right) \ge tf(\vec{x}) + (1-t)f(\vec{y})$$

- g is a convex function on U if $\forall \vec{x}, \vec{y} \in U, t \in [0,1]$

$$g\left(t\vec{x} + (1-t)\vec{y}\right) \le tg(\vec{x}) + (1-t)g(\vec{y})$$

- A linear function h is both concave and convex