Second Order Conditions for Constrained Optimization

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Let f, h be C^2 functions on R^2. To maximize f on the constraint set $C_h = \big\{(x,y)\big|h(\vec{x}) = c\big\}$, Form the Lagrangian

$$\mathcal{L}(\vec{x},\vec{\mu}) = f(\vec{x}) - \mu[h(\vec{x}) - c].$$

(x^*, y^*, μ^*) is a local constrained max of f on C_h if there exists μ^* such that (x^*, y^*, μ^*) satisfies:

1. FOC:

$$\frac{\partial \mathcal{L}}{\partial x} = 0, \ \ \frac{\partial \mathcal{L}}{\partial y} = 0, \ \ \frac{\partial \mathcal{L}}{\partial \mu} = 0 \text{ at } (x^*, y^*, \mu^*)$$

2. Bordered Matrix satisfies:

$$\det \begin{pmatrix} 0 & \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial^2 \mathcal{L}}{\partial x^2} & \frac{\partial^2 \mathcal{L}}{\partial x \partial y} \\ \frac{\partial h}{\partial y} & \frac{\partial^2 \mathcal{L}}{\partial x \partial y} & \frac{\partial^2 \mathcal{L}}{\partial y^2} \end{pmatrix} > 0 \text{ at } (x^*, y^*, \mu^*)$$

Complete the following questions:

1. Suppose $\dfrac{\partial h}{\partial x}(x^*, y^*) \neq 0$. How does Simon and Blume (1994) prove the above Theorem?

2. Now suppose $\dfrac{\partial h}{\partial x}(x^*, y^*) = 0$. Write your own proof assuming instead that $\dfrac{\partial h}{\partial y}(x^*, y^*) \neq 0$.

3. Prove the inequality version of the above theorem:

Let f, g be C^2 functions on R^2. To maximize f on the constraint set $C_g = \Big\{(x, y)\Big|g(\vec{x}) \leq b\Big\}$, Form the Lagrangian

$$\mathcal{L}(\vec{x}, \vec{\mu}) = f(\vec{x}) - \lambda[g(\vec{x}) - b].$$

Then, (x^*, y^*, λ^*) is a local constrained max of f on C_g if there exists λ^* such that (x^*, y^*, λ^*) satisfies:

(a) FOC:

$$\frac{\partial \mathcal{L}}{\partial x} = 0, \quad \frac{\partial \mathcal{L}}{\partial y} = 0, \quad \lambda^* \geq 0, \quad \lambda^*[g(\vec{x} - b] = 0 \text{ at } (x^*, y^*, \mu^*)$$

(b) Suppose g is binding at (x^*, y^*), then Bordered Matrix satisfies:

$$\det \begin{pmatrix} 0 & \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial^2 \mathcal{L}}{\partial x^2} & \frac{\partial^2 \mathcal{L}}{\partial x \partial y} \\ \frac{\partial g}{\partial y} & \frac{\partial^2 \mathcal{L}}{\partial x \partial y} & \frac{\partial^2 \mathcal{L}}{\partial y^2} \end{pmatrix} > 0 \text{ at } (x^*, y^*, \lambda^*)$$

Otherwise, if g is not binding, the Hessian of the Lagrangian is negative definite, or

$$\frac{\partial^2 \mathcal{L}}{\partial x^2} < 0, \text{ and } \det \begin{pmatrix} \frac{\partial^2 \mathcal{L}}{\partial x^2} & \frac{\partial^2 \mathcal{L}}{\partial x \partial y} \\ \frac{\partial^2 \mathcal{L}}{\partial x \partial y} & \frac{\partial^2 \mathcal{L}}{\partial y^2} \end{pmatrix} > 0 \text{ at } (x^*, y^*, \lambda^*).$$