到目前為止，我們有：

1. Equality Constraint：Thm 18.1
2. Several Equality Constraints：Thm 18.2
3. Inequality Constraint：Thm 18.3

因此自然也有：

4. Several Inequality Constraints：Thm 18.4

Thm 18.4 Suppose $f, g_1, \cdots, g_k \in C^1$ (functions of n variables)

Let $x^* \in \mathbb{R}^n$ solves

$$\begin{aligned} \text{Max } & f(x_1, \cdots, x_n) \\ \text{s.t. } & g_1(x_1, \cdots, x_n) \le b_1 \\ & \vdots \\ & g_k(x_1, \cdots, x_n) \le b_k \end{aligned}$$

Notation: At x^*, $1 \sim k_0$ constraints bind: $g_1(x_1^*, \cdots, x_n^*) = b_1, \cdots, g_{k_0}(x_1^*, \cdots, x_n^*) = b_{k_0}$

$(k_0+1), \cdots, k$ constraints not bind: $g_{k_0+1}(x_1^*, \cdots, x_n^*) < b_{k_0+1}, \cdots, g_k(x_1^*, \cdots, x_n^*) < b_k$

The Jacobian matrix of the binding constraints

[NDCQ] $$\begin{pmatrix} Dg_1 \\ \vdots \\ Dg_{k_0} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x^*) & \cdots & \frac{\partial g_1}{\partial x_n}(x^*) \\ \vdots & & \vdots \\ \frac{\partial g_{k_0}}{\partial x_1}(x^*) & \cdots & \frac{\partial g_{k_0}}{\partial x_n}(x^*) \end{pmatrix}$$ has rank k_0

i.e. vectors $Dg_1, \cdots, Dg_{k_0}$ are linearly independent

For $\mathcal{L} = f(x) - \lambda_1[g_1(x_1,\cdots,x_n) - b_1] - \lambda_2[g_2(x_1,\cdots,x_n) - b_2] - \cdots - \lambda_k[g_k(x_1,\cdots,x_n) - b_k]$

Then, $\exists\ \lambda^* = (\lambda_1^*, \cdots, \lambda_k^*)$ such that

(a) $\frac{\partial \mathcal{L}}{\partial x_1}(x^*, \lambda^*) = 0$, $\frac{\partial \mathcal{L}}{\partial x_2}(x^*, \lambda^*) = 0$, $\cdots$, $\frac{\partial \mathcal{L}}{\partial x_n}(x^*, \lambda^*) = 0$

(b) $\lambda_1^* \cdot [g_1(x^*) - b_1] = 0$, $\lambda_2^* \cdot [g_2(x^*) - b_2]$, $\cdots$, $\lambda_k^* \cdot [g_k(x^*) - b_k] = 0$

(c) $\lambda_1^* \geqslant 0$, $\lambda_2^* \geqslant 0$, $\cdots$, $\lambda_k^* \geqslant 0$

(d) $g_1(x^*) \leq b_1$, $g_2(x^*) \leq b_2$, $\cdots$, $g_k(x^*) \leq b_k$.

Exercise: 你可以寫出合併 Thm 18.2 & Thm 18.4 的 Mixed constraints 版本嗎?

(See 18.4)

Exercise: Thm 18.4 如何推廣到求 min ?

(See 18.5)

Question: 經濟學通常只考慮 $x_1 \geqslant 0$, $x_2 \geqslant 0$, $\cdots$, $x_n \geqslant 0$ 的情況.
但是每次都多 n 條 inequality 很麻煩. 有辦法簡化嗎?

(This is the Kuhn-Tucker version of Lagrange Method)

Thm 18.7 (Kuhn-Tucker Conditions)

For $\quad \text{Max } f(x_1, \cdots, x_n)$

$$\text{s.t. } g_1(x_1, \cdots, x_n) \le b_1, \cdots, g_k(x_1, \cdots, x_n) \le b_k$$

$$x_1 \geq 0, \cdots, x_n \geq 0$$

Form $\tilde{\mathcal{L}} = f(x) - \lambda_1 [g_1(x) - b_1] - \cdots - \lambda_k [g_k(x) - b_k]$

Suppose x^* is a solution and $\left(\frac{\partial g_i}{\partial x_j}\right)$ has maximum rank where $i \in \{i \mid g_i(x^*) = b_i\}$ for the binding constraints and $j \in \{j \mid x_j^* > 0\}$ for positive x_j^*. (i.e. vectors $D_j g_i$ are linearly independent)

Then, $\exists\ \lambda_1^*, \cdots, \lambda_k^* \geq 0$ such that

$$\boxed{A} \begin{cases} \dfrac{\partial \tilde{\mathcal{L}}}{\partial x_1} \le 0, & \underline{\text{with "=" if } x_1 > 0} \quad \Leftrightarrow x_1 \cdot \dfrac{\partial \tilde{\mathcal{L}}}{\partial x_1} = 0 \\ \vdots & \\ \dfrac{\partial \tilde{\mathcal{L}}}{\partial x_n} \le 0, & \underline{\text{with "=" if } x_n > 0}, \quad \Leftrightarrow x_n \cdot \dfrac{\partial \tilde{\mathcal{L}}}{\partial x_n} = 0 \end{cases}$$

$$\boxed{B} \begin{cases} \dfrac{\partial \tilde{\mathcal{L}}}{\partial \lambda_1} \geq 0, & \underline{\text{with "=" if } \lambda_1 > 0} \quad \Leftrightarrow \lambda_1 \cdot \dfrac{\partial \tilde{\mathcal{L}}}{\partial \lambda_1} = 0 \\ \vdots & \\ \dfrac{\partial \tilde{\mathcal{L}}}{\partial \lambda_k} \geq 0, & \underline{\text{with "=" if } \lambda_k > 0} \quad \Leftrightarrow \lambda_k \dfrac{\partial \tilde{\mathcal{L}}}{\partial \lambda_k} = 0 \end{cases}$$

(Proof) Use Thm 18.4 to show this...

$\Downarrow$

For $\mathcal{L} = f(x) - \lambda_1 [g_1(x_1, \cdots, x_n) - b_1] - \lambda_2 [g_2(x_1, \cdots, x_n) - b_2] - \cdots - \lambda_k [g_k(x_1, \cdots, x_n) - b_k]$

Then, $\exists\ \lambda^* = (\lambda_1^*, \cdots, \lambda_k^*)$ such that

(a) $\dfrac{\partial \mathcal{L}}{\partial x_1}(x^*, \lambda^*) = 0, \ \dfrac{\partial \mathcal{L}}{\partial x_2}(x^*, \lambda^*) = 0, \ \cdots, \ \dfrac{\partial \mathcal{L}}{\partial x_n}(x^*, \lambda^*) = 0$

(b) $\lambda_1^* \cdot [g_1(x^*) - b_1] = 0, \ \lambda_2^* \cdot [g_2(x^*) - b_2], \cdots, \lambda_k^* \cdot [g_k(x^*) - b_k] = 0$

(c) $\lambda_1^* \geq 0, \ \lambda_2^* \geq 0, \ \cdots, \ \lambda_k^* \geq 0$

(d) $g_1(x^*) \le b_1, \ g_2(x^*) \le b_2, \ \cdots, g_k(x^*) \le b_k$.

(Proof) Use Thm 18.4 to show this...

$$\mathcal{L} = f(x_1,\cdots,x_n) - \lambda_1\left[g_1(x_1,\cdots,x_n) - b_1\right] - \cdots - \lambda_k\left[g_k(x_1,\cdots,x_n) - b_k\right]$$
$$- \nu_1[-x_1 - 0] - \cdots - \nu_n(-x_n - 0)$$
$$= \tilde{\mathcal{L}} - \nu_1[-x_1 - 0] - \cdots - \nu_n(-x_n - 0)$$

FOC:

(a)
$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_1} = \dfrac{\partial f}{\partial x_1} - \lambda_1 \dfrac{\partial g_1}{\partial x_1} - \cdots - \lambda_k \dfrac{\partial g_k}{\partial x_1} + \nu_1 = 0 = \dfrac{\partial \tilde{\mathcal{L}}}{\partial x_1} + \nu_1 \\ \vdots \\ \dfrac{\partial \mathcal{L}}{\partial x_n} = \dfrac{\partial f}{\partial x_n} - \lambda_1 \dfrac{\partial g}{\partial x_n} - \cdots - \lambda_k \dfrac{\partial g_k}{\partial x_n} + \nu_n = 0 = \dfrac{\partial \tilde{\mathcal{L}}}{\partial x_n} + \nu_n \end{cases} \Rightarrow \begin{cases} \dfrac{\partial \tilde{\mathcal{L}}}{\partial x_1} = -\nu_1 \leq 0 \\ \vdots \\ \dfrac{\partial \tilde{\mathcal{L}}}{\partial x_n} = -\nu_n \leq 0 \end{cases}$$

(b)
$$\begin{cases} \lambda_1\left[g_1(x_1,\cdots,x_n) - b_1\right] = 0 \\ \vdots \\ \lambda_k\left[g_k(x_1,\cdots,x_n) - b_k\right] = 0 \end{cases} \qquad \left.\begin{cases} \nu_1 x_1 = 0 \\ \vdots \\ \nu_n x_n = 0 \end{cases}\right\} \Rightarrow \begin{cases} x_1 \cdot \dfrac{\partial \tilde{\mathcal{L}}}{\partial x_1} = 0 \\ \vdots \\ x_n \cdot \dfrac{\partial \tilde{\mathcal{L}}}{\partial x_n} = 0 \end{cases}$$

(c) $\lambda_1, \cdots, \lambda_n, \underline{\nu_1, \cdots, \nu_n} \geq 0$

(d)
$$\begin{cases} g_1(x_1,\cdots,x_n) \leq b_1, \\ \vdots \\ g_k(x_1,\cdots,x_n) \leq b_k \end{cases} \qquad \begin{cases} x_1 \geq 0 \\ \vdots \\ x_n \geq 0 \end{cases}$$

Hence we have $\boxed{A}$:

$$\frac{\partial \tilde{\mathcal{L}}}{\partial x_i} \leq 0 \quad \text{with "=" if } x_i > 0$$
$$\left(\text{i.e. } x_i \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial x_i} = 0\right)$$

(Proof) Use Thm 18.4 to show this...

$$\mathcal{L} = f(x_1,\cdots,x_n) - \lambda_1\left[g_1(x_1,\cdots,x_n) - b_1\right] - \cdots - \lambda_k\left[g_k(x_1,\cdots,x_n) - b_k\right]$$
$$- \nu_1[-x_1 - 0] - \cdots - \nu_n(-x_n - 0)$$
$$= \tilde{\mathcal{L}} - \nu_1[-x_1 - 0] - \cdots - \nu_n(-x_n - 0)$$

FOC:

(a)
$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_1} = \dfrac{\partial f}{\partial x_1} - \lambda_1 \dfrac{\partial g_1}{\partial x_1} - \cdots - \lambda_k \dfrac{\partial g_k}{\partial x_1} + \nu_1 = 0 \\ \vdots \\ \dfrac{\partial \mathcal{L}}{\partial x_n} = \dfrac{\partial f}{\partial x_n} - \lambda_1 \dfrac{\partial g}{\partial x_n} - \cdots - \lambda_k \dfrac{\partial g_k}{\partial x_n} + \nu_n = 0 \end{cases}$$

(b)
$$\left.\begin{cases} \lambda_1\left[g_1(x_1,\cdots,x_n) - b_1\right] = 0 \\ \vdots \\ \lambda_k\left[g_k(x_1,\cdots,x_n) - b_k\right] = 0 \end{cases}\right\} \qquad \begin{cases} \nu_1 x_1 = 0 \\ \vdots \\ \nu_n x_n = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 \cdot \dfrac{\partial \tilde{\mathcal{L}}}{\partial \lambda_1} = 0 \\ \vdots \\ \lambda_k \dfrac{\partial \tilde{\mathcal{L}}}{\partial \lambda_k} = 0 \end{cases}$$

(c) $\lambda_1, \cdots, \lambda_n, \nu_1, \cdots, \nu_n \geq 0$

(d)
$$\left.\begin{cases} g_1(x_1,\cdots,x_n) \leq b_1, \\ \vdots \\ g_k(x_1,\cdots,x_n) \leq b_k \end{cases}\right\} \qquad \begin{cases} x_1 \geq 0 \\ \vdots \\ x_n \geq 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial \tilde{\mathcal{L}}}{\partial \lambda_1} \leq 0, \quad \text{with "=" if } \lambda_1 > 0 \\ \vdots \\ \dfrac{\partial \tilde{\mathcal{L}}}{\partial \lambda_k} \leq 0, \quad \text{with "=" if } \lambda_k > 0 \end{cases} \Rightarrow \boxed{B}$$

#

Example: (Peak-Load Pricing) Assume p_1, p_2, c_1, c_2 & $C_0'(x_0) > 0$

$$\max\ \pi(x_1, x_2, x_0) = p_1 \cdot D_1(x_1) + p_2 \cdot D_2(x_2) - c_1 x_1 - c_2 x_2 - C_0(x_0)$$

$$\left.\begin{array}{ll} \text{s.t.} & x_1 \le x_0, \quad x_2 \le x_0 \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_0 \ge 0 \end{array}\right\} \Rightarrow \text{linear constraints NDCQ ok (except at } x_1 = x_2 = 0)$$

(Sol) $\tilde{\mathcal{L}} = p_1 \cdot D_1(x_1) - c_1 x_1 + p_2 \cdot D_2(x_2) - c_2 x_2 - C_0(x_0) - \lambda_1 [x_1 - x_0] - \lambda_2 [x_2 - x_0]$

FOC:

$$\frac{\partial \tilde{\mathcal{L}}}{\partial x_1} = p_1 \cdot D_1'(x_1) - c_1 - \lambda_1 \overset{\le}{=} 0 \quad \text{with "=" if } x_1 > 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial x_2} = p_2 D_2'(x_2) - c_2 - \lambda_2 \overset{\le}{=} 0 \quad \text{with "=" if } x_2 > 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial x_0} = \lambda_1 + \lambda_2 - C_0'(x_0) \overset{\le}{=} 0 \quad \text{with "=" if } x_0 > 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_1} = x_1 - x_0 \overset{\ge}{=} 0 \quad \text{with "=" if } \lambda_1 > 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_2} = x_2 - x_0 \overset{\ge}{=} 0 \quad \text{with "=" if } \lambda_2 > 0$$

(Continued)

$p_1 \cdot D_1'(x_1) - c_1 - \lambda_1 \overset{\le}{=} 0$ with "=" if $x_1 > 0$

$p_2 D_2'(x_2) - c_2 - \lambda_2 \overset{\le}{=} 0$ with "=" if $x_2 > 0$

$\lambda_1 + \lambda_2 - C_0'(x_0) \overset{\le}{=} 0$ with "=" if $x_0 > 0$ $\Rightarrow x_0 > 0$ (otherwise $x_1 = x_2 = 0$ $\Rightarrow \pi = 0$ not max.)

$x_1 - x_0 \overset{\ge}{=} 0$ with "=" if $\lambda_1 > 0$ — implies $\lambda_1 + \lambda_2 = C_0'(x_0) > 0$

$x_2 - x_0 \overset{\ge}{=} 0$ with "=" if $\lambda_2 > 0$ — $\Rightarrow \lambda_1 > 0$ or $\lambda_2 > 0$

$\lambda_1 > 0 \Downarrow x_1 = x_0 > 0$; $\lambda_2 > 0 \Downarrow x_2 = x_0 > 0$

Case 1: $\lambda_1 > 0 \Rightarrow x_1 = x_0 > 0 \Rightarrow p_1 D_1'(x_1) = c_1 + \lambda_1$ (MR=MC!) $\rightarrow x_1$ is peak!

Case 2: $\lambda_1 = 0 \Rightarrow \lambda_2 > 0 \Rightarrow x_2 = x_0 > 0 \Rightarrow p_2 D_2'(x_2) = c_2 + \lambda_2 \rightarrow x_2$ is peak!

(Could x_1, x_2 Both bind? Maybe! But at least one binds)

Example: Given Production cost $C(y) \in C^1$ satisfies $C' > 0$

Firm chouse output $y \in \mathbb{R}_+$, advertising cost $a \in \mathbb{R}_+$

to maximize Revenue $R(y,a) \in C^1$, $\frac{\partial R}{\partial a} > 0$, $R(0,a) = 0$

without letting profit drop below $m > 0$

(Sol) $\text{Max } R(y,a)$

s.t. $\pi = R(y,a) - C(y) - a \geq m$

$y \geq 0$, $a \geq 0$

$\mathcal{L} = R(y,a) - \lambda\left[m - R(y,a) + C(y) + a\right]$, If NDCQ holds,

FOC: $\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial R}{\partial y}(1+\lambda) - \lambda C'(y) \leq 0$, with "=" if $y > 0$

$\frac{\partial \mathcal{L}}{\partial a} = \frac{\partial R}{\partial a}(1+\lambda) - \lambda \leq 0$, with "=" if $a > 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = R(y,a) - C(y) - a - m \leq 0$, with "=" if $\lambda > 0$

(Continued) $\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial R}{\partial y}(1+\lambda) - \lambda C'(y) \leq 0$, with "=" if $y > 0$

$\frac{\partial \mathcal{L}}{\partial a} = \frac{\partial R}{\partial a}(1+\lambda) - \lambda \leq 0$, with "=" if $a > 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = R(y,a) - C(y) - a - m \leq 0$, with "=" if $\lambda > 0$

① If $\lambda = 0$, $\frac{\partial \mathcal{L}}{\partial a} = \frac{\partial R}{\partial a} > 0$ (contradiction!) Hence, $\lambda > 0$.

② $\lambda > 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \lambda} = 0$ i.e. $\pi = R(y,a) - C(y) - a = m > 0$ (minimum profit!)

③ If $y = 0$, $R(0,a) = 0$, not maximum (check later)

④ If $y > 0$, $\frac{\partial R}{\partial y}(1+\lambda) - \lambda C'(y) = 0 \Rightarrow \frac{\partial R}{\partial y} = \frac{\lambda}{1+\lambda} C'(y) > 0$

$\Rightarrow \frac{\partial \pi}{\partial y} = \frac{\partial R}{\partial y} - C'(y) = \frac{-C'(y)}{1+\lambda} < 0$

$\therefore y^* >$ profit-maximizing $\hat{y}$ since $\hat{y}$ requires $\frac{\partial \pi}{\partial y} = \frac{\partial R}{\partial y} - C'(\hat{y}) = 0$