Utility Representation - Notes

Utility Function Representation (Notes)

Preference For two consumption bundles $\begin{cases} x = (x_1, \cdots, x_n) \in \mathbb{R}^n \\ y = (y_1, \cdots, y_n) \in \mathbb{R}^n \end{cases}$

$x \succsim_h y \equiv$ " x is at least as good as y "

(這跟"10 ≥ 9"有何不同? Ans: 偏好要对 (2,1) & (1,2) 作出排序!)

Utility $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a utility function representing preference $\succsim_h$

if for all $x, y \in X \subset \mathbb{R}^n$, $u(x) \geq u(y) \iff x \succsim_h y$

(可以用效用函數 $u(x)$ 代表你的偏好 $\succsim_h$)

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EX: $X = \{(0,0), (0,1), (1,0), (1,1)\}$ ~ finite → $u(x)$ natural numbers

Easy: $u(0,0) = 0$, $u(0,1) = 1$, $u(1,0) = 2$, $u(1,1) = 3$

EX: $X = \{(x_1, x_2) : x_1 \in \mathbb{N}, x_2 \in \mathbb{N}\} = \{(1,1), (1,2), (2,1), (3,1), (2,2), (1,3), \cdots\}$

$(1,1) \to 1$, $(1,2) \to \frac{1}{2}$, $(2,1) \to 2$, $(3,1) \to 3$, $(2,2) \to 1$, $(1,3) \to \frac{1}{3}$

(X countable, need $u(x) \in \mathbb{Q} = \{x = \frac{p}{q} ; p, q \in \mathbb{Z}\}$)

What if X is a convex subset of $\mathbb{R}^n$?

Def: X is convex if for all $x^0, x^1 \in X$, $x^\lambda = (1-\lambda)x^0 + \lambda x^1 \in X$.

Utility Function Representation of Preferences

If preference $\succsim_h$ is complete, transitive, continuous on $X \subset \mathbb{R}^n$ and strictly monotonic, then it can be locally represented by a function $U(X)$ which is continuous over X.

- Complete: For all $x_1, x_2 \in X$, either $x_1 \succsim_h x_2$ or $x_2 \succsim_h x_1$

- Transitive: For $x_1, x_2, x_3 \in X$, if $x_1 \succsim_h x_2$, $x_2 \succsim_h x_3$, then $x_1 \succsim_h x_3$

- Continuous: Suppose $\{x^t\}_{t=1,2,\cdots} \longrightarrow x^0$, For any $y \in X$

  If $\begin{cases} x^t \succsim_h y \\ y \succsim_h x^t \end{cases}$ for all t, then $\begin{cases} x^0 \succsim_h y \\ y \succsim_h x^0 \end{cases}$,

- Strictly Monotonic: If $y \geq x$, then $y \succ_h x$.

  - vector inequality:
  - strictly prefer. $y \succsim_h x$ & $x \not\succsim_h y$

$y \geqslant x$ iff $y_i \geqslant x_i$ for all i

$y \gg x$ iff $y_i > x_i$ for all i

$y > x$ iff $y_i \geqslant x_i$ for all i, and $y_k > x_k$ for some k.

Utility Function Representation of Preferences

If preference $\succsim_h$ is complete, transitive, continuous on $X \subset \mathbb{R}^n$ and strictly monotonic, then it can be locally represented by a function $U(X)$ which is continuous over X.

(pf) Consider $x^0, x' \in X$ such that $x' > x^0$ $(\Rightarrow x' \succ_h x^0)$

For $T = \{x \in X \mid x' \succsim_h \underline{x} \succsim_h x^0\} \subset X \subset \mathbb{R}^n$

Claim:

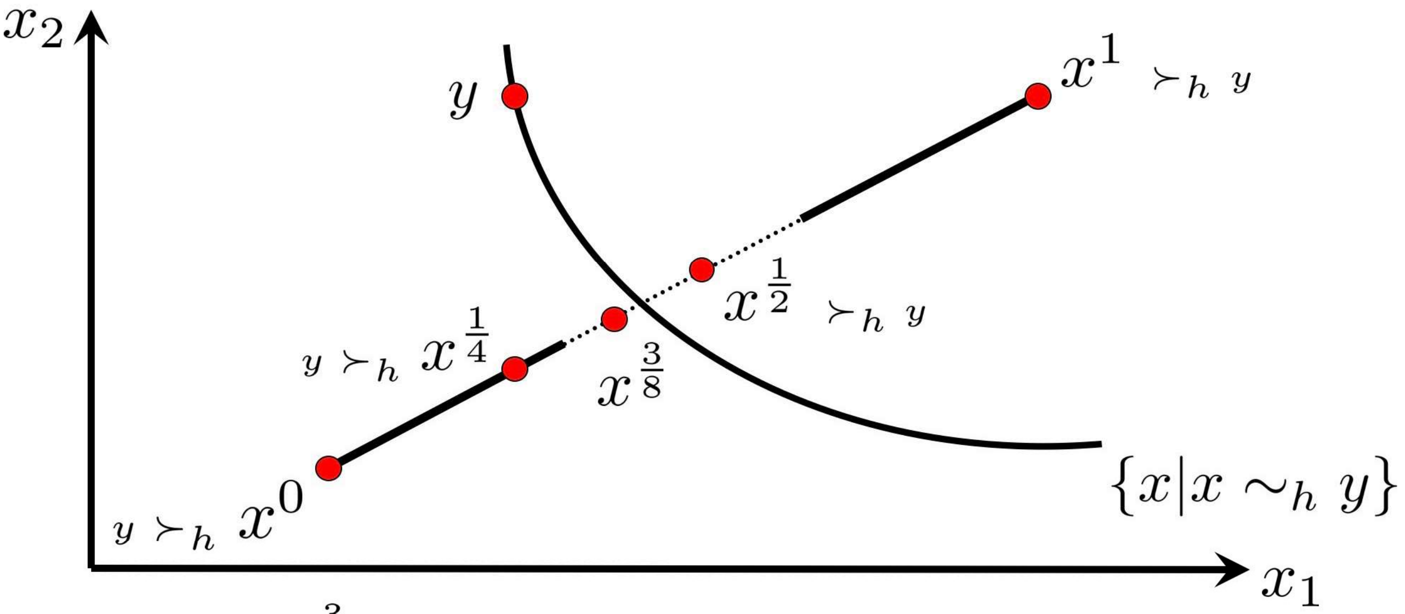
[A] For any $y \in T$, there exists some weight $\lambda \in [0, 1]$ such that

$$y \sim_h x^\lambda = (1-\lambda)x^0 + \lambda x'$$

[B] Moreover, $\lambda(y): T \longrightarrow [0, 1]$ is continuous.

(See Graph)

Special Case: Strict Monotonicity



Either $x^{\frac{3}{8}} \sim_h y$ (done),
$x^{\frac{3}{8}} \succ_h y$ (consider $x^{\frac{3}{16}}$), or $y \succ_h x^{\frac{3}{8}}$ (consider $x^{\frac{7}{16}}$).

(Proof of [A]) We appeal to the completeness of real numbers:

Consider the sequences $\{\nu_t\}$, $\{\mu_t\}$ starting from $\nu_0 = 1$, $\mu_0 = 0$.

Let $\lambda_t = \frac{\nu_t + \mu_t}{2}$, define ν_{t+1}, μ_{t+1} as: (If $y \sim_h x^{\nu_0}$ or $y \sim_h x^{\mu_0}$ then done.)

1. If $y \sim_h x^{\lambda_t}$, done.
2. If $y \succ_h x^{\lambda_t}$, let $\nu_{t+1} = \nu_t$, $\mu_{t+1} = \lambda_t$
3. If $x^{\lambda_t} \succ_h y$, let $\nu_{t+1} = \lambda_t$, $\mu_{t+1} = \mu_t$

Hence, we have

$$x^1 = x^{\nu_0} \succsim_h \cdots \succsim_h x^{\nu_n} \succsim_h y \succsim_h x^{\mu_n} \succsim_h \cdots \succsim_h x^{\mu_0} = x^0$$

and sequence of interval $\{[\mu_t, \nu_t]\}_{t=0,1,2,\cdots}$ such that $\nu_t - \mu_t \to 0$

By Completeness of $\mathbb{R}$, there exists $\lambda \in [\mu_t, \nu_t]$ for all t. 而且唯一 (unique)

(區間套一定會套中!)

Thus, we have (By continuity)

$$\{x^{\nu_t}\}_{t=0,1,2,\dots} \longrightarrow x^{\lambda} \quad \text{such that } x^{\nu_t} \succ_h y \Rightarrow x^{\lambda} \succsim_h y$$

$$\{x^{\mu_t}\}_{t=0,1,2,\dots} \longrightarrow x^{\lambda} \quad \text{such that } y \succ_h x^{\mu_t} \Rightarrow y \succsim_h x^{\lambda}$$

Or, $y \sim x^{\lambda}$

Exercise: Show that $\lambda(y)$ indeed represent $\succsim_h$ on T

i.e. $\lambda(y_1) \geq \lambda(y_2) \iff y_1 \succsim_h y_2$ for all $y_1, y_2 \in T$.

What about $\boxed{B}$?

Exercise:

(1) Show that if $\lambda(y)$ is discontinuous at y, there exists $\hat{\varepsilon} > 0$ and $\{y^t\}_{t=1}^{\infty} \longrightarrow y$ such that $|\lambda(y^t) - \lambda(y)| > \hat{\varepsilon}$.

(2) Consider 2 subsequences $\begin{cases} \{y_+^t\}_{t=1}^{\infty} & \text{such that } \lambda(y_+^t) - \lambda(y) > \hat{\varepsilon} \\ \{y_-^t\}_{t=1}^{\infty} & \text{such that } \lambda(y_-^t) - \lambda(y) < -\hat{\varepsilon} \end{cases}$

Show that at least one is infinite and violates continuity of preferences.