Infinite Sequences and Series

Section 11.1-11.2

Outline

- Sequences:
  - Definition
  - The Limit of a Sequence
  - Properties of Limits
  - Monotonic Sequence Theorem
- Series:
  - Definition
  - Examples
  - Properties

Sequences

- Definition: A **sequence** is a list of infinite numbers written in a definite order:
  $a_1, a_2, a_3, \cdots, a_n, \cdots$
- Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as *a function whose domain is the set of positive integers*.
- Notation: The sequence $\{a_1, a_2, a_3, \cdots\}$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Subsequences

- Given a sequence $\{a_n\}$, we say that $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ and denote it by $\{a_{n_k}\} \subset \{a_n\}$ if $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$ i.e. $\{n_k\}$ is an increasing sequence of positive integers.
- $\{a_{2n}\}$ are the even terms of $\{a_n\}$.
- $\{a_{2n+1}\}$ are the odd terms of $\{a_n\}$.

The Limit of a Sequence

- Definition: A sequence $\{a_n\}$ has the limit L and we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$ if we can make the terms a_n as close to L as we like by taking n sufficiently large.
- If $\lim_{n\to\infty} a_n = L$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

The Limit of a Sequence

- Definition:
- $\lim_{n\to\infty} a_n = L$ if for every $\epsilon > 0$ there is a corresponding integer N such that if $n > N$ then $|a_n - L| < \epsilon$.

The Limit of a Sequence

- Definition:
  $\lim_{n\to\infty} a_n = \infty$ means that for every $M > 0$ there is an integer N such that if $n > N$ then $a_n > M$.
- Theorem: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for $n \in N$, then $\lim_{n\to\infty} a_n = L.$
- Example: Find $\lim_{n\to\infty} r^n$, and $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}$.

Properties of Limits

- Limit Laws for Sequences:
- If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c\, a_n = c \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \text{ if } \lim_{n\to\infty} b_n \neq 0$$

Properties of Limits

- Squeeze Theorem for Sequences:
- If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.
- Example: Find $\lim_{n\to\infty} \frac{n!}{n^n}$.

Properties of Limits

- Theorem: $\lim_{n\to\infty} |a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$.
- Theorem:
- If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.
- Example:
  $\lim_{n\to\infty} a_n^p = (\lim_{n\to\infty} a_n)^p$ if $p > 0$ and $a_n > 0$.

Properties of Limits

- Theorem:
- If a sequence $\{a_n\}$ converges to the limit L, then every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ converges to L .
- Theorem:
- Given a sequence $\{a_n\}$, if both $\{a_{2n}\}$ and $\{a_{2n+1}\}$ converge to L, then $\lim_{n\to\infty} a_n = L$.

Monotonic Sequence Theorem

- Monotonic Sequence Theorem:
- Every bounded, monotonic sequence is convergent.
- Definition: A sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \geq 1$. $\{a_n\}$ is decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. $\{a_n\}$ is monotonic if it is either increasing or decreasing.

Monotonic Sequence Theorem

- Definition:
- A sequence $\{a_n\}$ is bounded above if there is a number M such that $a_n \le M$ for all $n \ge 1$.
  $\{a_n\}$ is bounded below if there is a number m such that $m \le a_n$ for all $n \ge 1$.
- If $\{a_n\}$ is bounded above and below, then it is a bounded sequence.

Monotonic Sequence Theorem

- The proof of Monotonic Sequence Theorem is based on the **Completeness Axiom** for the set of real numbers, which says that :
  *"If S is a nonempty set of real numbers that has an upper bound M (x ≤ M for all x in S), then S has a **least upper bound** b."*
- Definition: b is the *least upper bound* of S if
  - 1. b is an upper bound for S.
  - 2. $b \le M$ for any other upper bound M.

Least Upper Bound

- Notations for least upper bound of S:
  $l.u.b.\ S$ or $\sup S$.
- Properties of the least upper bound:
- If S has a least upper bound b, then b is unique. Moreover, if $a < b$, then a is not an upper bound for S and there is some $x \in S$ such that $a < x \le b$.

Remark

- Sometimes we formulate the completeness axiom of real numbers as " Nested Interval Property".
- Nested Interval Property:
- If $I_n = [a_n, b_n]$, $n \in N$, is a nested sequence of closed bounded intervals (i.e. $I_{n+1} \subseteq I_n$ for all $n \ge 1$) then there is a real number ξ such that $\xi \in I_n$ for all $n \ge 1$(i.e. $\xi \in \cap_{n=1}^{\infty} I_n$).

Remark:

- At this point, we see that the properties of real numbers serve as the foundation of Calculus. Mathematicians like to focus on these properties, and take the real numbers as undefined objects satisfying certain axioms from which all the other properties will be derived.
- The axioms of real numbers fall into three groups which are the *field axioms*, the *order axioms* and the *completeness axiom*.

Remark:

- For further study about the axioms of real numbers, you could read the first chapter of any book about the advanced calculus. For example,

  Mathematical Analysis, Apostol, Addison Wesley.

Series

- Definition: When we try to add the terms of an infinite sequence, we get an expression of the form $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ which is called an **infinite series** (or just a **series**) and is denoted, by the symbol $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.
- We first add up first n terms, and let s_n denote the nth partial sum $\sum_{i=1}^{n} a_i$.

Series

- Definition:
- $\sum a_n$ is **convergent** if the sequence $\{s_n\}$ is convergent. If $\lim_{n\to\infty} s_n = s$, we write $\sum_{n=1}^{\infty} a_n = s$ and call s the **sum** of the series.
- If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Series: Examples

- The **geometric series**. For $a \neq 0$, consider

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

- It is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad .$$

Series: Examples

- The **harmonic series**:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$ is divergent.

- Actually, we can show that $S_{2^n} > 1 + \frac{n}{2}$, or $S_n > \ln(n+1)$ for $n > 1$.

Properties of Series

- Theorem: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.
- The converse of the theorem is not true in general! But we can derive the following test.
- Test for Divergence: If $\lim_{n\to\infty} a_n$ doesn't exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Properties of Series

- Theorem: If $\sum a_n$ and $\sum b_n$ are convergent series, then $\sum c\, a_n$ (where c is a constant), $\sum (a_n + b_n)$, and $\sum (a_n - b_n)$ are also convergent. Moreover,

$$\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

Summary

- What is an infinite sequence? What is the limit of a sequence?
- Review properties of limits of sequences.
- State the Monotonic Sequence Theorem.
- State the Completeness Axiom for real numbers.
- What is a series? What is its sum?
- Review the geometric series and the harmonic series.