Calculus 4 With Applications in Economics and Management – Final Exam

PART A: True or False

Determine whether the following statements are True or False:

1. (2%) Every bounded nonempty set of rational numbers has a least upper bound which is also a rational number.

2. (2%) If $\{a_n\}$ is a bounded increasing sequence, then $\{a_n\}$ converges to the least upper bound of $\{a_n\}$.

3. (2%) If b is the least upper bound of S, a subset of real numbers, then for every $\epsilon > 0$, there is an $s \in S$ such that $b - \epsilon < s \leq b$.

4. (2%) Every bounded sequence has a convergent subsequence.

5. (2%) Suppose that $f(x, y)$ is continuous on $\mathbb{R}^2$ and $f(x_0, y_0) = 0$, $f(x_1, y_1) = 1$. Let $p_0 = (x_0, y_0)$ and $p_1 = (x_1, y_1)$. Then, for every $0 < \lambda < 1$, there is some (x_λ, y_λ) on the line segment $\overline{p_0 p_1}$ such that $f(x_\lambda, y_\lambda) = \lambda$. (Ans: FTTTT)

PART B: (15%) Find the interval of convergence of the power series

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{4^n \cdot n \cdot \ln n}(x-3)^n.$$

Ans: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n \ln n}{4(n+1)\ln(n+1)}|x-3| = \frac{n}{n+1} \cdot \frac{\ln n}{\ln(n+1)} \cdot \frac{|x-3|}{4} \to \frac{|x-3|}{4}$ as $n \to \infty$.

By the ratio test, if $\frac{|x-3|}{4} < 1$, $\sum_{n=2}^{\infty} a_n$ converges absolutely. If $\frac{|x-3|}{4} > 1$, then $\sum_{n=2}^{\infty} a_n$ diverges. Therefore, the radius of convergence is 4.

For $x - 3 = 4$, $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n}$. Consider the function $f(x) = \frac{1}{x \ln x}$, $f(x)$ is positive, continuous and decreasing on $[2, \infty)$ and $f(n) = a_n$. Hence, by the integral test, $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges if and only if $\int_2^{\infty} f(x)dx$ converges. Therefore, $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges since

$$\int_2^{\infty} f(x)dx = \lim_{T\to\infty} \int_2^T f(x)dx = \lim_{T\to\infty} \ln(\ln(x))\Big|_{x=2}^{T} = \lim_{T\to\infty} \ln(\ln T) - \ln(\ln 2) = \infty.$$

For $x - 3 = -4$, $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \cdot \ln n}$. Since $\left\{\frac{1}{n \ln n}\right\}$ is positive, decreasing and $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$, the alternating series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges. Thus, the power series converges for $x \in [-1, 7)$.

PART C: Consumer Theory

Consider a consumer who enjoys n goods $\vec{x} = (x_1, \cdots, x_n)$, and has the utility function $U(x_1, \cdots, x_n) = -\sum_{i=1}^{n} a_i(x_i - b_i)^2$, $a_i > 0$, $b_i > 0$, which is defined on $x_1 \geq 0, \cdots, x_n \geq 0$. We assume the consumer has income I to spend, and faces market price $\vec{p} = (p_1, \cdots, p_n)$. Assuming I, p_1, $\cdots$ and $p_n > 0$, consumer's budget constraint is $\displaystyle\sum_{i=1}^{n} p_i x_i \leq I$.

1. (5%) State the Kuhn-Tucker version Lagrangian function and its first order conditions.

   Ans: $$\tilde{\mathcal{L}}(x_1, \cdots, x_n, \lambda) = -\sum_{i=1}^{n} a_i(x_i - b_i)^2 - \lambda\left(\sum_{i=1}^{n} p_i x_i - I\right)$$

   The first order conditions are

   $$\begin{aligned}
   \frac{\partial \tilde{\mathcal{L}}}{\partial x_i} &= -2a_i(x_i - b_i) - \lambda p_i \leq 0, \quad x_i \geq 0, \text{ for } 1 \leq i \leq n \\
   x_i \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial x_i} &= x_i \cdot (-2a_i x_i + a_i b_i - \lambda p_i) = 0 \text{ for } 1 \leq i \leq n \\
   \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda} &= I - \sum_{i=1}^{n} p_i x_i \geq 0, \quad \lambda \geq 0, \\
   \lambda \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda} &= \lambda\left(I - \sum_{i=1}^{n} p_i x_i\right) = 0
   \end{aligned}$$

2. (10%) Now suppose $\displaystyle\sum_{i=1}^{n} p_i b_i > I$. Is there a $\vec{x}^*(\vec{p}, I) = \big(x_1^*(p_1, \cdots, p_n, I), \cdots, x_n^*(p_1, \cdots, p_n, I)\big)$ with $x_i^* > 0$ for $i = 1, \cdots, n$ that satisfies the first order conditions? Find such $\vec{x}^*(\vec{p}, I)$. Note that $\vec{x}^*(\vec{p}, I)$ maximizes utility subject to the budget constraint, so it is called the demand function.

   Ans: If $(x_1^*, \cdots, x_n^*)$ satisfies the above first order conditions and $x_i > 0$ for $1 \leq i \leq n$, then from $x_i \cdot \dfrac{\partial \tilde{\mathcal{L}}}{\partial x_i} = 0$ we derive $-2a_i x_i + 2a_i b_i - \lambda p_i = 0$. Hence, $x_i = b_i - \lambda \cdot \dfrac{p_i}{2a_i}$ for $1 \leq i \leq n$. If $\lambda = 0$, then $x_i = b_i$, but then $I - \displaystyle\sum_{i=1}^{n} p_i b_i < 0$ violating the budget constraint. Hence, $\lambda > 0$. So, $I = \displaystyle\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i\left(b_i - \lambda \cdot \frac{p_i}{2a_i}\right)$, or $\lambda \cdot \left(\displaystyle\sum_{i=1}^{n} \frac{p_i^2}{2a_i}\right) = \sum_{i=1}^{n} p_i b_i - I$.

   Thus, $\lambda = \dfrac{\sum_{i=1}^{n} p_i b_i - I}{\sum_{i=1}^{n} \frac{p_i^2}{2a_i}} > 0$, and we have derived $x_i^* = b_i - \dfrac{\sum_{i=1}^{n} p_i b_i - I}{\sum_{i=1}^{n} \frac{p_i^2}{2a_i}} \cdot \dfrac{p_i}{2a_i}$ for $1 \leq i \leq n$.

3. (5%) (Continued) Find the maximized utility $V(\vec{p}, I) = \max\left\{U(x_1, \cdots, x_n) \middle| \sum_{i=1}^{n} p_i x_i \le I\right\}$.

Ans: $V(\vec{p}, I) = U(x_1^*, \cdots, x_n^*) = -\sum_{i=1}^{n} a_i (x_i^* - b_i)^2$

$$= -\sum_{i=1}^{n} a_i \cdot \frac{p_i^2}{4a_i^2} \cdot \left(\frac{\sum_{j=1}^{n} p_j b_j - I}{\sum_{j=1}^{n} \frac{p_j^2}{2a_j}} \right)^2 = -\frac{\left(\sum_{j=1}^{n} p_j b_j - I\right)^2}{\sum_{j=1}^{n} \frac{p_j^2}{a_j}}.$$

4. (5%) Use Envelope Theorem to derive $\frac{\partial V}{\partial I}(\vec{p}, I)$ and $\frac{\partial V}{\partial p_i}(\vec{p}, I)$. What is the relationship between $\frac{\frac{\partial V}{\partial p_i}(\vec{p}, I)}{\frac{\partial V}{\partial I}(\vec{p}, I)}$ and the demand function?

Ans: By the Envelope Theorem, we have

$$\frac{\partial V}{\partial I}(\vec{p}, I) = \frac{\partial \mathcal{L}}{\partial I} = \frac{\partial \tilde{\mathcal{L}}}{\partial I} = \lambda^*, \quad \frac{\partial V}{\partial p_i}(\vec{p}, I) = \frac{\partial \mathcal{L}}{\partial p_i} = \frac{\partial \tilde{\mathcal{L}}}{\partial p_i} = -\lambda^* x_i^*.$$

Hence, $-\frac{\frac{\partial V}{\partial p_i}}{\frac{\partial V}{\partial I}} = \frac{\lambda^* x_i^*}{\lambda^*} = x_i^*(\vec{p}, I)$. This is called the Roy's identity in microeconomic theory.

5. (bonus) What is the maximum achievable utility $U^{\max}$ for all possible $x_i \ge 0$ and $I \ge 0$? What is the minimum $U^{\min}$?

Ans: $U^{\max} = 0$ at $x_i = b_i > 0$. At $I = 0$, we have $U^{\min} = -\sum_{i=1}^{n} a_i b_i^2$.

6. (bonus) For all feasible $\overline{U} \in [U^{\min}, U^{\max}]$, solve for the expenditure function $M(\vec{p}, \overline{U}) = \min\left\{\sum_{i=1}^{n} p_i x_i \middle| U(x_1, \cdots, x_n) \ge \overline{U}\right\}$. (Hint: Use what you already know from above!)

Ans: Note that $V\left(\vec{p}, M(\vec{p}, \overline{U})\right) = \overline{U}$ for $V(\vec{p}, I) = -\frac{\left(\sum_{j=1}^{n} p_j b_j - I\right)^2}{\sum_{j=1}^{n} \frac{p_j^2}{a_j}}$. This is called duality.

Hence, we have $\overline{U} = -\frac{\left(\sum_{j=1}^{n} p_j b_j - M(\vec{p}, \overline{U})\right)^2}{\sum_{j=1}^{n} \frac{p_j^2}{a_j}}$, or $M(\vec{p}, \overline{U}) = \sum_{j=1}^{n} p_j b_j - \sqrt{(-\overline{U}) \sum_{j=1}^{n} \frac{p_j^2}{a_j}}$.

Also note that we can derive compensated demand using Envelope Theorem:

$$\frac{\partial M}{\partial p_i} = x^c(\vec{p}, \overline{U}) = b_i + \frac{\frac{\overline{U} p_i}{a_i}}{\sqrt{(-\overline{U}) \sum_{j=1}^{n} \frac{p_j^2}{a_j}}}.$$

PART D: Joint Production

Suppose a monopoly farm breeds q_0 chicken to produce q_1 chicken drumsticks and q_2 chicken breasts. $q_0, q_1, q_2 \in \mathbb{R}, q_0, q_1, q_2 \geq 0$. Since each chicken has two legs and one breast, output $\vec{q} = (q_1, q_2)$ cannot exceed production constraints $q_1 \leq 2q_0$ and $q_2 \leq q_0$. Breeding chicken q_0 requires a fixed cost $F = 10,000$ and constant marginal cost $c_0 = 200$, and selling each product requires a constant marginal cost of packaging, $\vec{c} = (c_1, c_2) = (5, 10)$. Hence, the farm's total cost is

$$C(q_0, \vec{q}) = F + c_0 q_0 + c_1 q_1 + c_2 q_2 = 10,000 + 200q_0 + 5q_1 + 10q_2.$$

Let the demand function for each product depend on consumption of both products:

$$p_1 = p_1(\vec{q}) = p_1(q_1, q_2) = 955 - \frac{1}{3}q_1^2 - q_2$$
$$p_2 = p_2(\vec{q}) = p_2(q_1, q_2) = 320 - q_1 - q_2$$

1. (5%) Write down the profit-maximization problem for this farm.

   Ans: Since total revenue is $R(q_0, q_1, q_2) = p_1(q_1, q_2) \cdot q_1 + p_2(q_1, q_2) \cdot q_2$, the firm solves:

$$\max \pi(q_0, q_1, q_2) = \left(955 - \frac{1}{3}q_1^2 - q_2\right) \cdot q_1 + (320 - q_1 - q_2) \cdot q_2 - (10,000 + 200q_0 + 5q_1 + 10q_2)$$
$$\text{s. t. } g_1(q_0, q_1, q_2) = q_1 - 2q_0 \leq 0$$
$$g_2(q_0, q_1, q_2) = q_2 - q_0 \leq 0$$
$$q_0 \geq 0, q_1 \geq 0, q_2 \geq 0$$

   Note that q_0, q_1, q_2 are continuous variables, instead of discrete. This is of course unrealistic, but can be a good approximation, especially when quantities are large.

2. (10%) State the Kuhn-Tucker version Lagrangian. Is the corresponding NDCQ satisfied?

   Ans: $\tilde{\mathcal{L}}(q_0, q_1, q_2) = \pi(q_0, q_1, q_2) - \lambda_1 g_1(q_0, q_1, q_2) - \lambda_2 g_2(q_0, q_1, q_2)$

$$= 955q_1 - \frac{1}{3}q_1^3 - q_1q_2 + 320q_2 - q_1q_2 - q_2^2 - 10,000 - 200q_0 - 5q_1 - 10q_2 - \lambda_1(q_1 - 2q_0) - \lambda_2(q_2 - q_0)$$

   The Kuhn-Tucker NDCQ requires full rank for $\left(\frac{\partial g_i}{\partial q_j}\right)(\vec{q}^*, \vec{\lambda}^*)$ over binding g_i and $q_j > 0$. If $q_0^* = 0$, then $q_1^* = q_2^* = 0$. Therefore, the matrix is empty and NDCQ is trivially satisfied. If $q_0^* > 0$ and g_i is binding, then either $q_1^* = q_0^*$ or $q_2^* = 2q_0^*$, which means that $q_i^* > 0$. Hence, the matrix $\left(\frac{\partial g_i}{\partial q_j}\right)$ has the terms $\frac{\partial g_i}{\partial q_0}, \frac{\partial g_i}{\partial q_i}$ depending on which binds. Since gradients are

$$\vec{\nabla} g_1 = \left(\frac{\partial g_1}{\partial q_0}, \frac{\partial g_1}{\partial q_1}, \frac{\partial g_1}{\partial q_2}\right) = (-2, 1, 0)$$
$$\vec{\nabla} g_2 = \left(\frac{\partial g_2}{\partial q_0}, \frac{\partial g_2}{\partial q_1}, \frac{\partial g_2}{\partial q_2}\right) = (-1, 0, 1),$$

   NDCQ is indeed satisfied regardless of which conatraints binds: When only $q_1 > 0$, $(-2, 1)$ and $(-1, 0)$ are linearly independent. When only $q_2 > 0$, $(-2, 0)$ and $(-1, 1)$ are linearly independent. When both $q_1, q_2 > 0$, the two gradients above are linearly independent.

3. (5%) State the corresponding first order conditions.

Ans: The first order conditions are

$$\frac{\partial \tilde{\mathcal{L}}}{\partial q_0} = -200 + 2\lambda_1 + \lambda_2 \le 0, \quad q_0 \ge 0, \qquad q_0 \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial q_0} = q_0 \cdot (-200 + 2\lambda_1 + \lambda_2) = 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial q_1} = 955 - q_1^2 - 2q_2 - 5 - \lambda_1 \le 0, \quad q_1 \ge 0, \qquad q_1 \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial q_1} = q_1 \cdot (955 - q_1^2 - 2q_2 - 5 - \lambda_1) = 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial q_2} = 320 - 2q_1 - 10 - \lambda_2 \le 0, q_2 \ge 0, \qquad q_2 \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial q_2} = q_2 \cdot (310 - 2q_1 - \lambda_2) = 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_1} = 2q_0 - q_1 \ge 0, \quad \lambda_1 \ge 0 \qquad \lambda_1 \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_1} = \lambda_1 (2q_0 - q_1) = 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_2} = q_0 - q_2 \ge 0, \quad \lambda_2 \ge 0 \qquad \lambda_2 \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_2} = \lambda_2 (q_0 - q_2) = 0$$

4. Consider the case of $q_1^* < 2q_0^*$. First assume $q_0^* > 0$.

   (a) (15%) Is there a set of (q_0^*, q_1^*, q_2^*) satisfying the first order conditions under this case?

   Ans: If $q_0^* > 0$, $q_1^* < 2q_0^*$, then $-200 + 2\lambda_1 + \lambda_2 = 0$. Since $\lambda_1^*(2q_0^* - q_1^*) = 0$, we have $\lambda_1^* = 0$ and $\lambda_2^* = 200$. Hence, $q_0^* = q_2^* > 0$. Therefore, the first order condition of q_2 becomes $310 - 2q_1^* - 2q_2^* - \lambda_2^* = 110 - 2q_1^* - 2q_2^* = 0$.

   If $q_1^* = 0$, then $q_2^* = \frac{110}{2} = 55 = q_0^*$ and the first order condition of q_1 becomes $950 - (q_1^*)^2 - 2q_2^* - \lambda_1^* = 950 - 110 > 0$. But this contradicts FOC ≤ 0, so we conclude that $q_1^* > 0$. Thus, the first order condition of q_1 becomes $950 - (q_1^*)^2 - 2q_2 = 0$.

   Combining the two equations, we have $110 - 2q_1^* = 2q_2^* = 950 - (q_1^*)^2$. Therefore, $(q_1^*)^2 - 2q_1^* - 840 = 0 = (q_1^* - 30)(q_1^* + 28)$. Hence, $q_1^* = 30$ (since $-28 < 0$), and $q_2^* = \frac{1}{2}(110 - 2 \cdot 30) = 25 = q_0^*$. Thus, $(q_0^*, q_1^*, q_2^*, \lambda_1^*, \lambda_2^*) = (25, 30, 25, 0, 200)$ satisfies all first order conditions.

   (b) (10%) Check second order conditions at this (q_1^*, q_2^*, q_0^*). Is it a local maximum, local minimum, or saddle point?

   Ans: Consider the matrix

$$\begin{pmatrix} 0 & \vec{\nabla} g_2 \\ (\vec{\nabla} g_2)^T & \left(\frac{\partial^2 \tilde{\mathcal{L}}}{\partial q_i \partial q_j}\right) \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -2q_1 & -2 \\ 1 & 0 & -2 & -2 \end{pmatrix}.$$

   Since only $g_2(q_0, q_1, q_2) = q_2 - q_0$ is binding and $n = 3$, we need to check the last two leading principle minors at $(25, 30, 25, 0, 200)$

$$\det \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -60 & -2 \\ 1 & 0 & -2 & -2 \end{pmatrix} = -116, \text{ and } \det \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -60 \end{pmatrix} = 60.$$

Since the determinant of the entire matrix has the same sign as $(-1)^n = (-1)^3$ and the last two leading principle minors alternate in sign, we conclude that $(25, 30, 25)$ is a local maximum.

(c) (5%) Verify that the maximized profit at (q_1^*, q_2^*, q_0^*) is indeed larger than the profit if one chooses $q_0^{**} = 0$ (to rule out this case).

Ans: $\pi(25, 30, 25) = (955-300-25)\cdot 30+(320-55)\cdot 25-(10,000+5000+150+250) = 630 \cdot 30 + 265 \cdot 25 - 15,400 = 18,900 + 6,625 - 15,400 = 10,125$.

When $q_0^* = 0$, $q_1^* = q_2^* = 0$. Then, $\pi(0, 0, 0) = -10,000 < \pi(25, 30, 25)$. Hence, $q_0^{**} = 0$ does not maximize profit π.

5. (bonus) Show that there is no (q_1^*, q_2^*, q_0^*) satisfying the first order conditions if $q_2^* < q_0^*$.
   Ans: Exercise.

6. (bonus) Are there other possibilities? What is the solution to this maximization problem?
   Ans: Exercise.