Calculus 4 With Applications in Economics and Management – Quiz 2

Name

Student ID

Consider a consumer who enjoys three goods (x, y, z), and has the utility function $u(x, y, z) = x$ subject to the constraints $y \geq x$ and $z \geq x$. This utility function is define on $x \geq 0$, $y \geq 0$, $z \geq 0$. We assume the consumer has income I to spend, and faces market price P_x, P_y and P_z for goods x, y, z, respectively. Assuming I, P_x, P_y, $P_z > 0$, consumer's budget constraint is

$$P_x x + P_y y + P_z z \leq I.$$

1. (1%) Write down the maximization problem for this consumer.

2. (2%) State the Kuhn-Tucker version Lagrangian. Is the corresponding NDCQ satisfied?

3. (2%) Write down the first order conditions for this problem.

4. (2%) What is the utility-maximizing (x^*, y^*, z^*) in terms of I, P_x, P_y, and P_z?

Now suppose $I = \$22,000$, $P_x = \$100$, $P_y = \$200$ and $P_z = \$10$.

5. (2%) How should income I be allocated among goods x, y and z to yield the largest possible utility?

6. (2%) Use the Envelope Theorem to estimate the change in maximized utility if income I decreased by \$1,000.

7. (2%) Compute the exact change in the previous question.

8. (2%) Use the Envelope Theorem to estimate the change in maximized utility if P_z, the price of goods z, drops from \$10 to \$5.

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Suggested Answers

1. (1%) Write down the maximization problem for this consumer.

   $$(\textbf{Sol.})\quad \max\left\{f(x,y,z)=x \;\middle|\; x\le y, x\le z,\; P_x x+P_y y+P_z z\le I, x\ge 0, y\ge 0, z\ge 0\right\}$$

2. (2%) State the Kuhn-Tucker version Lagrangian. Is the corresponding NDCQ satisfied?

   $$(\textbf{Sol.})\quad \tilde{\mathcal{L}}(x,y,z,\lambda_1,\lambda_2,\lambda_3)=x-\lambda_1(x-y)-\lambda_2(x-z)-\lambda_3(P_x x+P_y y+P_z z-I)$$

   Let $g_1(x,y,z)=x-y$, $g_2(x,y,z)=x-z$, $g_3(x,y,z)=P_x x+P_y y+P_z z$. Obviously, $x=0$ cannot obtain maximum $f(x,y,z)=x$ since we can choose a $x_0>0$, $y_0>0$, $z_0>0$ such that (x_0,y_0,z_0) satisfies all the constraints (given $I>0$) and $f(x_0,y_0,z_0)=x_0>0$. Hence, if (x^*,y^*,z^*) is the solution, $x^*>0$, and $y^*\ge x^*>0$, $z^*\ge x^*>0$.

   For NDCQ, we need to find the binding constraints $\{g_i\}$ at (x^*,y^*,z^*) and check whether its gradients $\{\vec{\nabla}g_i(x^*,y^*,z^*)\}$ are linearly independent. Here, if we can show that

   $$\vec{\nabla}g_1=(1,-1,0),\quad \vec{\nabla}g_2=(1,0,-1),\quad \vec{\nabla}g_3=(P_x,P_y,P_z),$$

   are linearly independent, any subset of $\left\{\vec{\nabla}g_1,\vec{\nabla}g_2,\vec{\nabla}g_3\right\}$ is linearly independent. In fact, if

   $$a_1\left[\vec{\nabla}g_1\right]+a_2\left[\vec{\nabla}g_2\right]+a_3\left[\vec{\nabla}g_3\right]=(0,0,0)=(a_1+a_2+a_3P_x,-a_1+a_3P_y,-a_2+a_3P_z),$$

   then $a_1=a_3P_y$, $a_2=a_3P_z$, and $a_1+a_2+a_3P_x=a_3(P_x+P_y+P_z)=0$. $P_x,P_y,P_z>0$ implies $a_3=0$, and $a_1=a_3P_y=0$, $a_2=a_3P_z=0$, so $\left\{\vec{\nabla}g_1,\vec{\nabla}g_2,\vec{\nabla}g_3\right\}$ are linearly independent.

3. (2%) Write down the first order conditions for this problem.

   (Sol.) For $x\ge 0$, $y\ge 0$, $z\ge 0$, $\lambda_1\ge 0$, $\lambda_2\ge 0$, $\lambda_3\ge 0$, FOC requires

   $$\begin{aligned}
   &\frac{\partial\tilde{\mathcal{L}}}{\partial x}=1-\lambda_1-\lambda_2-\lambda_3P_x\le 0, && x\cdot\frac{\partial\tilde{\mathcal{L}}}{\partial x}=0\\
   &\frac{\partial\tilde{\mathcal{L}}}{\partial y}=\lambda_1-\lambda_3P_y\le 0, && y\cdot\frac{\partial\tilde{\mathcal{L}}}{\partial y}=0\\
   &\frac{\partial\tilde{\mathcal{L}}}{\partial z}=\lambda_2-\lambda_3P_z\le 0, && z\cdot\frac{\partial\tilde{\mathcal{L}}}{\partial z}=0\\
   &\frac{\partial\tilde{\mathcal{L}}}{\partial \lambda_1}=y-x\ge 0, && \lambda_1\cdot\frac{\partial\tilde{\mathcal{L}}}{\partial \lambda_1}=\lambda_1(y-x)=0\\
   &\frac{\partial\tilde{\mathcal{L}}}{\partial \lambda_2}=z-x\ge 0, && \lambda_2\cdot\frac{\partial\tilde{\mathcal{L}}}{\partial \lambda_2}=\lambda_2(z-x)=0\\
   &\frac{\partial\tilde{\mathcal{L}}}{\partial \lambda_3}=I-P_x x-P_y y-P_z z\ge 0, && \lambda_3\cdot\frac{\partial\tilde{\mathcal{L}}}{\partial \lambda_3}=\lambda_3(I-P_x x-P_y y-P_z z)=0
   \end{aligned}$$

4. (2%) What is the utility-maximizing (x^*, y^*, z^*) in terms of I, P_x, P_y, and P_z?

(Sol.) In the discussion of NDCQ, we obtained $x^*, y^*, z^* > 0$, so $\frac{\partial \tilde{\mathcal{L}}}{\partial x} = 0$, $\frac{\partial \tilde{\mathcal{L}}}{\partial y} = 0$, $\frac{\partial \tilde{\mathcal{L}}}{\partial z} = 0$.

(a) Suppose $x^* < y^*$, then multiplier $\lambda_1 = 0$, and by $\frac{\partial \tilde{\mathcal{L}}}{\partial y} = 0$ we have $\lambda_3 = 0$ as well. Then, by $\frac{\partial \tilde{\mathcal{L}}}{\partial z} = 0$, we have $\lambda_2 = 0$. However, $\frac{\partial \tilde{\mathcal{L}}}{\partial x} = 0$ cannot be satisfied if $\lambda_1 = \lambda_2 = \lambda_3 = 0$. This is a contradiction, so the assumption $x^* < y^*$ cannot be true. Thus, $x^* = y^*$.

(b) Similiarly, suppose $x^* < z^*$. Then, multiplier $\lambda_2 = 0$, and by $\frac{\partial \tilde{\mathcal{L}}}{\partial z} = 0$ we have $\lambda_3 = 0$ as well. Then, by $\frac{\partial \tilde{\mathcal{L}}}{\partial y} = 0$, we have $\lambda_1 = 0$. However, $\frac{\partial \tilde{\mathcal{L}}}{\partial x} = 0$ cannot be satisfied if $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus, $x^* = z^*$.

(c) Since $x^* = y^* = z^* > 0$, from $\frac{\partial \tilde{\mathcal{L}}}{\partial x} = 0$, $\frac{\partial \tilde{\mathcal{L}}}{\partial y} = 0$ and $\frac{\partial \tilde{\mathcal{L}}}{\partial z} = 0$ we conclude that $\lambda_1 = \lambda_3 P_y$, $\lambda_2 = \lambda_3 P_z$, and $1 = \lambda_1 + \lambda_2 + \lambda_3 P_x = \lambda_3(P_y + P_z + P_x)$. Hence,

$$\lambda_3^* = \frac{1}{P_x + P_y + P_z} > 0, \quad \lambda_1^* = \frac{P_y}{P_x + P_y + P_z}, \quad \lambda_2^* = \frac{P_z}{P_x + P_y + P_z}, \quad \text{and} \quad \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_3} = 0.$$

Thus, we have $x^* = y^* = z^* = \dfrac{I}{P_x + P_y + P_z}$.

Now suppose $I = \$22,000$, $P_x = \$100$, $P_y = \$200$ and $P_z = \$10$.

5. (2%) How should income I be allocated among goods x, y and z to yield the largest possible utility?

(Sol.) $x^* = y^* = z^* = \dfrac{I}{P_x + P_y + P_z} = \dfrac{22,000}{310} \approx 70.97$, so among income $I = \$22,000$,

$$P_x x^* = \frac{P_x I}{P_x + P_y + P_z} = \frac{2,200,000}{310} \approx \$7,097 \text{ should be allocated goods } x,$$
$$P_y y^* = \frac{P_y I}{P_x + P_y + P_z} = \frac{4,400,000}{310} \approx \$14,194 \text{ should be allocated goods } y, \text{ and}$$
$$P_z z^* = \frac{P_z I}{P_x + P_y + P_z} = \frac{220,000}{310} \approx \$709.7 \text{ should be allocated goods } z.$$

6. (2%) Use the Envelope Theorem to estimate the change in maximized utility if income I decreased by \$1,000.

(Sol.) The Envelope Theorem states $\dfrac{d}{dI}\left[f(x^*(I), y^*(I), z^*(I), I)\right] = \dfrac{\partial \tilde{\mathcal{L}}}{\partial I} = \lambda_3^*$. Hence, for $\lambda_3^* = \dfrac{1}{P_x + P_y + P_z} = \dfrac{1}{310}$, the change in maximal utility is $\lambda_3^* \cdot \Delta I = \dfrac{-1000}{310} \approx -3.2258$.

7. (2%) Compute the exact change in the previous question.

(Sol.) If income $\tilde{I} = \$21,000$, then $x^{**} = y^{**} = z^{**} = \dfrac{\tilde{I}}{P_x + P_y + P_z} = \dfrac{21,000}{310} \approx 67.74$. Thus, $(x^{**} - x^*) = \dfrac{21,000}{310} - \dfrac{22,000}{310} = \dfrac{-100}{31} \approx -3.2258$ is the exact change in maximal utility. Note that this answer is exactly the same as the previous answer obtained by the Envelope Theorem since income enters the Lagrangian linearly.

8. (2%) Use the Envelope Theorem to estimate the change in maximized utility if P_z, the price of goods z, drops from \$10 to \$5.

(Sol.) The Envelope Theorem states $\frac{d}{dP_z}\left[f(x^*(P_z), y^*(P_z), z^*(P_z), P_z)\right] = \frac{\partial \tilde{\mathcal{L}}}{\partial P_z} = -\lambda_3^* z^*$. Hence, for $\lambda_3^* = \frac{1}{P_x + P_y + P_z} = \frac{1}{310}$ and $z^* = \frac{I}{P_x + P_y + P_z} = \frac{22,000}{310}$, the change in maximal utility is $-\lambda_3^* z^* \cdot \Delta P_z = -\frac{1}{310} \cdot \frac{22,000}{310} \cdot (-5) = \frac{1,100}{31^2} \approx 1.1446$.