Constrained Optimization and Kuhn-Tucker Conditions

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(Calculus 4, 18.4)

Theorem 18.4 (Several Inequality Constraints)

- Suppose f, $g_1, \ldots, g_k$ be C^1 functions on $\boldsymbol{R}^n$
- Let $\vec{x}^* = (x_1^*, \cdots, x_n^*)$ solve max. problem

$$\max\left\{f(x_1,\cdots,x_n)\middle|g_1(x_1,\cdots,x_n) \le b_1,\right.$$
$$\left.\cdots, g_k(x_1,\cdots,x_n) \le b_k\right\}$$

- Notation: Constraints $g_1, \ldots, g_{k_0}$ binds

$$g_1(x_1^*,\cdots,x_n^*) = b_1, \cdots, g_{k_0}(x_1^*,\cdots,x_n^*) = b_{k_0}$$

- Constraints $g_{k_0+1}, \ldots, g_k$ do not binds

$$g_{k_0+1}(x_1^*,\cdots,x_n^*) < b_{k_0+1}, \cdots, g_k(x_1^*,\cdots,x_n^*) < b_k$$

Theorem 18.4 (Several Inequality Constraints)

- Binding constraints $g_1, \ldots, g_{k_0}$ satisfies NDCQ if its Jacobian matrix has maximum rank k_0

$$\begin{pmatrix} \nabla g_1 \\ \vdots \\ \nabla g_{k_0} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\vec{x}^*) & \cdots & \frac{\partial g_1}{\partial x_n}(\vec{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{k_0}}{\partial x_1}(\vec{x}^*) & \cdots & \frac{\partial g_{k_0}}{\partial x_n}(\vec{x}^*) \end{pmatrix}$$

- Or, row vectors

$$Dg_i = \nabla g_i = \left(\frac{\partial g_i}{\partial x_1}(\vec{x}^*), \cdots, \frac{\partial g_i}{\partial x_n}(\vec{x}^*) \right)$$

are linearly independent

Theorem 18.4 (Several Inequality Constraints)

- Row vectors

$$Dg_i = \nabla g_i = \left(\frac{\partial g_i}{\partial x_1}(\vec{x}^*), \cdots, \frac{\partial g_i}{\partial x_n}(\vec{x}^*) \right)$$

are linearly independent if

$$a_1 \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\vec{x}^*) \\ \vdots \\ \frac{\partial g_1}{\partial x_n}(\vec{x}^*) \end{pmatrix} + \cdots + a_{k_0} \begin{pmatrix} \frac{\partial g_{k_0}}{\partial x_1}(\vec{x}^*) \\ \vdots \\ \frac{\partial g_{k_0}}{\partial x_n}(\vec{x}^*) \end{pmatrix} = \vec{0}$$

implies $a_1 = \cdots = a_{k_0} = 0$

Theorem 18.4 (Several Inequality Constraints)

For $\mathcal{L} = f(x_1, \cdots, x_n) - \lambda_1[g_1(x_1, \cdots, x_n) - b_1] - \cdots - \lambda_k[g_k(x_1, \cdots, x_n) - b_k]$

- There exists $\vec{\lambda}^* = (\lambda_1^*, \cdots, \lambda_k^*)$ such that

a) $\dfrac{\partial \mathcal{L}}{\partial x_1}(\vec{x}^*, \vec{\lambda}^*) = 0, \cdots, \dfrac{\partial \mathcal{L}}{\partial x_n}(\vec{x}^*, \vec{\lambda}^*) = 0$

b) $\lambda_1^*[g_1(\vec{x}^*) - b_1] = 0, \cdots, \lambda_k^*[g_k(\vec{x}^*) - b_k] = 0$

c) $\lambda_1^* \geq 0, \cdots, \lambda_k^* \geq 0$

d) $g_1(\vec{x}^*) - b_1 \leq 0, \cdots, g_k(\vec{x}^*) - b_k \leq 0$

Theorem 18.7 (Kuhn-Tucker)

- Suppose $f,\ g_1,\dots,\ g_k$ be C^1 functions on $\boldsymbol{R}^n$
- Let $\vec{x}^* = (x_1^*, \cdots, x_n^*)$ solve max. problem

$$\max\left\{ f(x_1, \cdots, x_n) \middle| x_1 \geq 0, \cdots, x_n \geq 0, g_1(x_1, \cdots, x_n) \leq b_1, \cdots, g_k(x_1, \cdots, x_n) \leq b_k \right\}$$

- NDCQ satisfied if $\left(\dfrac{\partial g_i}{\partial x_j}\right)_{ij}$ has maximum rank

Binding constraints Positive x_j

where $i \in \left\{ i \middle| g_i(\vec{x}^*) = b_i \right\}, j \in \left\{ j \middle| x_j^* > 0 \right\}$

- Exists $\vec{\lambda}^* = (\lambda_1^*, \cdots, \lambda_k^*), \lambda_i^* \geq 0$, such that

Theorem 18.7 (Kuhn-Tucker)

For $\tilde{\mathcal{L}} = f(x_1, \cdots, x_n) - \lambda_1[g_1(x_1, \cdots, x_n) - b_1] - \cdots - \lambda_k[g_k(x_1, \cdots, x_n) - b_k]$

A. $\dfrac{\partial \tilde{\mathcal{L}}}{\partial x_1}(\vec{x}^*, \vec{\lambda}^*) \leq 0, \cdots, \dfrac{\partial \tilde{\mathcal{L}}}{\partial x_n}(\vec{x}^*, \vec{\lambda}^*) \leq 0$

$x_1^* \cdot \dfrac{\partial \tilde{\mathcal{L}}}{\partial x_1}(\vec{x}^*, \vec{\lambda}^*) = 0, \cdots, x_n^* \cdot \dfrac{\partial \tilde{\mathcal{L}}}{\partial x_n}(\vec{x}^*, \vec{\lambda}^*) = 0$

B. $\dfrac{\partial \tilde{\mathcal{L}}}{\partial \lambda_1}(\vec{x}^*, \vec{\lambda}^*) \geq 0, \cdots, \dfrac{\partial \tilde{\mathcal{L}}}{\partial \lambda_k}(\vec{x}^*, \vec{\lambda}^*) \geq 0$

$\lambda_1^* \cdot \dfrac{\partial \tilde{\mathcal{L}}}{\partial \lambda_1}(\vec{x}^*, \vec{\lambda}^*) = 0, \cdots, \lambda_k^* \cdot \dfrac{\partial \tilde{\mathcal{L}}}{\partial \lambda_k}(\vec{x}^*, \vec{\lambda}^*) = 0$

Theorem 18.7 (Kuhn-Tucker)

- Let $x_1^* > 0, \cdots, x_{n_0}^* > 0,\ x_{n_0+1}^* = \cdots = x_n^* = 0$
- Binding constraints $g_1, \ldots, g_{k_0}$ satisfies NDCQ if the following matrix has maximum rank k_0

$$\begin{pmatrix} \hat{D}g_1 \\ \vdots \\ \hat{D}g_{k_0} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\vec{x}^*) & \cdots & \frac{\partial g_1}{\partial x_{n_0}}(\vec{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{k_0}}{\partial x_1}(\vec{x}^*) & \cdots & \frac{\partial g_{k_0}}{\partial x_{n_0}}(\vec{x}^*) \end{pmatrix}$$

- Or, row vectors

$$\hat{D}g_i = \left(\frac{\partial g_i}{\partial x_1}(\vec{x}^*), \cdots, \frac{\partial g_i}{\partial x_{n_0}}(\vec{x}^*) \right)$$

(1^{st} n_0 elements of ∇g_i) are linearly independent

Theorem 18.7 (Kuhn-Tucker)

- Row vectors (1st n_0 elements of ∇g_i)

$$\hat{D}g_i = \left(\frac{\partial g_i}{\partial x_1}(\vec{x}^*), \cdots, \frac{\partial g_i}{\partial x_{n_0}}(\vec{x}^*) \right)$$

are linearly independent if

$$a_1 \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\vec{x}^*) \\ \vdots \\ \frac{\partial g_1}{\partial x_{n_0}}(\vec{x}^*) \end{pmatrix} + \cdots + a_{k_0} \begin{pmatrix} \frac{\partial g_{k_0}}{\partial x_1}(\vec{x}^*) \\ \vdots \\ \frac{\partial g_{k_0}}{\partial x_{n_0}}(\vec{x}^*) \end{pmatrix} = \vec{0}$$

implies $a_1 = \cdots = a_{k_0} = 0$

Exercise 18.14 (Generalize Example 18.9)

$$\begin{aligned} \max \quad & f(x,y,z) = xyz \\ \text{s.t.} \quad & P_x x + P_y y + P_z z \leq I \\ & x \geq 0, y \geq 0, z \geq 0 \end{aligned}$$

- NDCQ?

$$\tilde{\mathcal{L}} = xyz - \lambda[P_x x + P_y y + P_z z - I]$$

- FOC?

Exercise 18.14 (Generalize Example 18.9)

$$\max \quad f(x, y, z) = xyz$$

$$\text{s.t.} \quad P_x x + P_y y + P_z z \leq I$$

$$x \geq 0, y \geq 0, z \geq 0$$

- NDCQ?

$$\tilde{\mathcal{L}} = xyz - \lambda[P_x x + P_y y + P_z z - I]$$

Exercise 18.14 (Generalize Example 18.9)

$$\tilde{\mathcal{L}} = xyz - \lambda[P_x x + P_y y + P_z z - I]$$

FOC:

$$\frac{\partial \tilde{\mathcal{L}}}{\partial x} = yz - \lambda P_x \leq 0, x \cdot \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial y} = xz - \lambda P_y \leq 0, y \cdot \frac{\partial \mathcal{L}}{\partial y} = 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial z} = xy - \lambda P_z \leq 0, z \cdot \frac{\partial \mathcal{L}}{\partial z} = 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \lambda} = I - (P_x x + P_y y + P_z z) \geq 0, \lambda \cdot \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda} = 0$$

Exercise 18.14 (Generalize Example 18.9)

$$\tilde{\mathcal{L}} = xyz - \lambda[P_x x + P_y y + P_z z - I]$$

Solution:

$$x^* = \frac{I}{3P_x}$$

$$y^* = \frac{I}{3P_y}$$

$$z^* = \frac{I}{3P_z}$$

$$f(x^*, y^*, z^*) = \frac{I^3}{27 P_x P_y P_z}$$

Ex: Sales-Maximizing Firm with Advertising

- Suppose $R(y, a)$, $C(y)$ are C^1 functions satisfying $C'(y)>0$, $R(0, a)=0$, $\frac{\partial R}{\partial a} > 0$
- Firms choose y, a from $\boldsymbol{R}_+$ to maximize revenue $R(y, a)$, without letting profit drop below $m > 0$

$$\max_{y,a} \; R(y,a)$$

$$\text{s.t. } \Pi = R(y,a) - C(y) - a \geq m$$

$$y \geq 0, a \geq 0$$

Ex: Sales-Maximizing Firm with Advertising

- Suppose $C'(y)>0$, $R(0, a)=0$, $\dfrac{\partial R}{\partial a} > 0$

$$
\begin{aligned}
\max_{y,a} \quad & R(y,a) \\
\text{s.t.} \quad & \Pi = R(y,a) - C(y) - a \geq m \\
& y \geq 0, a \geq 0
\end{aligned}
$$

1. Show that the constraint binds, so the firm will maintain minimum profit
2. Show that output (if positive) is larger than profit-maximizing output

Ex: Sales-Maximizing Firm with Advertising

- Wait, does NDCQ always hold? No!

$$\max_{y,a} \; R(y,a)$$

$$\text{s.t.} \; \Pi = R(y,a) - C(y) - a \geq m$$

$$y \geq 0, a \geq 0$$

$$g(a,y) = m - R(y,a) + C(y) + a$$

$$\nabla g = \left(\frac{\partial g}{\partial y}, \frac{\partial g}{\partial a}\right) = \left(-\frac{\partial R}{\partial y} + C'(y), -\frac{\partial R}{\partial a} + 1\right)$$

$= 0$ if MR $=$ MC and advertising MR $= 1$

The Meaning of the Multiplier

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(Calculus 4, 19.1)

Theorem 19.1 (Single Equality Constraint)

- Consider $\max_{x,y} \left\{ f(x,y) \middle| h(x,y) = a \right\}$
- Let f, h be continuously differentiable (C^1)
- For any fixed value a, let $(x^*(a), y^*(a), \mu^*(a))$ be the solution which satisfies NDCQ.
  - (Implicit Function Theorem applies!)
- Suppose x^*, y^*, μ^* are C^1 functions of a
- Then, $\mu^*(a) = \frac{d}{da} f\left(x^*(a), y^*(a)\right)$

Theorem 19.2 (Several Equality Constraints)

For $\max\left\{f(x_1,\cdots,x_n)\middle|h_1(x_1,\cdots,x_n)=a_1,\right.$

$\left.\cdots,h_m(x_1,\cdots,x_n)=a_m\right\}$

- Let f, $h_1,\ldots,$ h_m be C^1 functions on $\boldsymbol{R}^n$
- For $\vec{a}=(a_1,\cdots,a_m)$, $x_1^*(\vec{a}),\cdots,x_n^*(\vec{a})$ is the solution with Lagrange Multipliers $\mu_1^*(\vec{a}),\cdots,\mu_m^*(\vec{a})$ which satisfies NDCQ
- Suppose x_i^*,μ_j^* are C^1 functions of $\vec{a}$, then

$$\frac{\partial f}{\partial a_j}(\vec{a})=\frac{\partial}{\partial a_j}f\left(x_1^*(\vec{a}),\cdots,x_n^*(\vec{a})\right)=\mu_j^*(\vec{a})\quad (j=1,\ldots,m)$$

Theorem 19.3 (Several Inequality Constraints)

For $\max\left\{f(x_1,\cdots,x_n)\middle|g_1(x_1,\cdots,x_n)\le a_1^*,\right.$

$\left.\cdots,g_k(x_1,\cdots,x_n)\le a_k^*\right\}$

- Let f, $g_1,\ldots,$ g_k be C^1 functions on $\boldsymbol{R}^n$
- For $\vec{a}^* = (a_1^*,\cdots,a_k^*)$, $x_1^*(\vec{a}^*),\cdots,x_n^*(\vec{a}^*)$ is the solution with Lagrange Multipliers $\lambda_1^*(\vec{a}^*),\cdots,\lambda_m^*(\vec{a}^*)$ which satisfies NDCQ
- Suppose x_i^*, λ_j^* are C^1 functions near $\vec{a}^*$, then

$$\frac{\partial f}{\partial a_j}(\vec{a}^*) = \frac{\partial}{\partial a_j} f\left(x_1^*(\vec{a}^*),\cdots,x_n^*(\vec{a}^*)\right) = \lambda_j^*(\vec{a}^*) \quad (j=1,\ldots,k)$$

Ex: Limited Resources, Profit-maximizing Firm

For $\max\left\{f(x_1,\cdots,x_n)\middle|g_1(x_1,\cdots,x_n)\le a_1^*,\right.$

$$\left.\cdots,g_k(x_1,\cdots,x_n)\le a_k^*\right\}$$

- Firm provide services *1,…,n* at levels $x_1,\ldots,x_n$
- To maximize profit $f(x_1,\ldots,x_n)$ by allocating inputs *1,…,k* at levels $g_1,\ldots,g_n$
  - Inputs *1,…,k* constrained by $\vec{a}^* = (a_1^*,\cdots,a_k^*)$
  - Addition profit for adding 1 more unit of input j

  = firm's WTP for adding 1 more unit of input j

  $= \lambda_j^*(\vec{a}^*)$

Exercise 18.14 (Generalize Example 18.9)

$$\max \quad f(x,y,z) = xyz$$

$$\text{s.t.} \quad P_x x + P_y y + P_z z \le I$$

$$x \ge 0, y \ge 0, z \ge 0$$

$$\tilde{\mathcal{L}} = xyz - \lambda[P_x x + P_y y + P_z z - I]$$

$$\Rightarrow f(x^*, y^*, z^*) = \frac{I^3}{27 P_x P_y P_z}$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial I}(x^*, y^*, z^*; I) = \lambda^* = \frac{y^* z^*}{P_x} = \frac{I^2}{9 P_x P_y P_z}$$

$$x^* = \frac{I}{3P_x}$$

$$y^* = \frac{I}{3P_y}$$

$$z^* = \frac{I}{3P_z}$$