(Econometric Theory III, September 2011, NTU)

Exercises for Practice: Linear Algebra

(1) Find the angle between the vectors $(1, 2, 0, 3)$ and $(2, 4, -1, 1)$.

(2) Find two unit vectors that are orthogonal to $(3, -2)$.

(3) Let S be a basis for an n-dimensional vector space V. Show that every set in V with more than n vectors must be linearly dependent.

(4) Let matrix A be symmetric. Show that $A^\top$ is symmetric.

(5) Show that orthogonal transformations preserve dot products and norms.

(6) Prove that a rotation matrix is an orthogonal matrix.

(7) Let X be an $n \times k$ matrix with $rank(X) = rank(X^\top X) = k < n$. Find $rank(X(X^\top X)^{-1}X^\top)$.

(8) Consider the quadratic form $f(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}$ such that A is not symmetric. Find $\nabla_{\mathbf{x}} f(\mathbf{x})$.

(9) Let X be an $n \times k$ matrix with full column rank and Σ be an $n \times n$ symmetric, positive definite matrix. Show that $X(X^\top \Sigma^{-1} X)^{-1} X^\top \Sigma^{-1}$ is a projection matrix but not an orthogonal projection matrix.

(10) Let ℓ be a vector of n ones. Show that $\ell\ell^\top/n$ is an orthogonal projection matrix.

(11) Let S_1 and S_2 be two subspaces of V such that $S_2 \subseteq S_1$. Let P_1 and P_2 be two orthogonal projection matrices projecting vectors onto S_1 and S_2, respectively. Find $P_1 P_2$ and $(I - P_1)(I - P_2)$.

(12) Show that a matrix is positive definite if and only if its eigenvalues are all positive.

(13) Let A be a symmetric and idempotent matrix. Show that $trace(A)$ is the number of non-zero eigenvalues of A and $rank(A) = trace(A)$.

(14) Let P be the orthogonal matrix such that $P^\top(A^\top A)P = \Lambda$, where A is $n \times k$ with rank $k < n$. What are the properties of $Z^* = AP$ and $Z = Z^*\Lambda^{-1/2}$? Note that the column vectors of Z^* (Z) are known as the (standardized) principal axes of $A^\top A$.